Square or Not? Assessing Constructions in an Interactive Geometry Software Environment

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Classroom geometry has gone "dynamic." A square that can be resized with a click and drag of a computer mouse holds enormous appeal for a generation accustomed to the static, hands-off nature of textbook illustrations. Activities incorporating interactive geometry software such as The Geometer's Sketchpad (Jackiw 2001) and Cabri Geometry (Laborde and Bellermain 1994) appear on a nearly monthly basis in the Mathematics Teacher, one of the classroom journals of the National Council of Teachers of Mathematics (NCTM). In 1997, the Mathematical Association of America published Geometry Turned On (King and Schattschneider 1997), an entire volume devoted to applications of interactive geometry software. Several secondary school geometry curricula include computer explorations in their texts (Gay 1998; Serra 2003), and Principles and Standards for School Mathematics (NCTM 2000) recommends the use of the software to promote mathematical investigations.

This interest in motion geometry is not new. Syer, writing in 1945, describes the ability of film to create "continuous" geometric images. His advocacy of the moving picture reads much as a modern-day justification for interactive geometry software (Syer 1945, p. 344):

In addition to true-to-life demonstrations of solid geometry, it would be interesting to make greater use of the peculiar advantages of moving pictures over ordinary models. In plane geometry films we used figures which changed shape, position, and color without distracting pauses or outside aid. This continuous and swift succession of illustrations is fast enough to keep

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Editor's note: The CD accompanying this yearbook contains one activity sheet and three Geometer's Sketchpad files that are relevant to this article. For more information on The Geometer's Sketchpad, contact Key Curriculum Press at www.keypress.com/.
up with a spoken description, or even as fast as the thought processes that are developing the idea. Thus no time is lost erasing pictures from the blackboard, changing lantern slides, or holding up illustrations, because the illustrations and thought move simultaneously.

Today’s software incorporates the motion available in film but goes one step further, providing users the tools to design their own animations. Exercising such control over a moving image places new demands on students, since they must learn how to build constructions that respond appropriately when dragged with the computer mouse. A well-behaved square, for example, will change its size and orientation when dragged, but not its shape.

How students negotiate the tools of interactive geometry software to accomplish their construction goals remains an open question in mathematics education literature (Goldenberg 1998). Are there instances where a particular software tool or technique impedes a student from completing a construction task (Laborde 1993)? Or do the software’s tools promote novel construction techniques that would not surface in a straightedge-and-compass environment?

This article describes a student-interview study conducted with E. Paul Goldenberg of Education Development Center that focused on the learning of geometry in an interactive geometry software environment. Middle school students received prebuilt “mystery” constructions on The Geometer’s Sketchpad that included such common geometric objects as squares, rectangles, isosceles triangles, and perpendicular bisector lines. Interviewees explored these objects by dragging each of their parts with the computer mouse (for a collection of similar activities, see Battista 2003).

As they experimented, students described what they observed on the screen and explained how they thought each object was built. Beginning then with a fresh blank screen, students attempted to reconstruct the identical objects from scratch. Throughout the interview, videotape recorded the mouse movements and menu selections of the students, as well as their accompanying commentary. The detailed nature of the tapes makes them an ideal source for analyzing the geometric conceptions of students in an interactive geometry setting.

### The Interview Setting and Procedures

Our interview study targeted middle school students with no prior interactive geometry experience. In total, we interviewed three sixth graders and five seventh graders from two middle schools (six boys and two girls). Each student participated in two interview sessions held on separate days, with individual sessions running approximately two hours. As the students explored and performed constructions with the software, one camera recorded the computer screen while another camera videotaped the interviewee. These separate images were transferred to a mixing board sitting outside the interview room, where a technician combined the shots to produce a split-screen composite tape.

As the students progressed through the interview tasks, the interviewer sat by their side and functioned in two roles:

1. When a student was uncertain whether the software contained a particular feature or forgot where it was located, the interviewer offered assistance. Throughout the session, the interviewer reminded students that the interview was not a test of how well they had memorized the software commands; rather, it was intended to uncover how they thought about the objects on screen.

2. From time to time, the interviewer would ask questions like, “What are you trying to do? Describe to me what you’re seeing. Can you explain why that line behaves the way it does? How might you test your theory?” The interviewer would also restate or rephrase some of the students' observations to spotlight ideas that would benefit from further attention.

### Constructing a Square

Compared to other geometric shapes, a square is a relatively simple object to construct with interactive geometry software, built with a circle tool and repeated applications of a perpendicular line command. The interview excerpts that follow, however, offer students construction techniques that depart from standard methods. With each technique comes the same question: Have the interviewees built a square? Although this may sound like a simple matter to answer, it is, in fact, a thornier issue than one might suppose in the world of interactive geometry.

Given a quadrilateral drawn on a piece of paper, it is easy enough to check if it fits the definition of a square—measure its sides and angles. If the sides are equal and the angles measure 90 degrees, the quadrilateral is a square. By contrast, an interactive geometry quadrilateral might pass these measurement tests and still not qualify as a square. To clarify the situation, Finzer and Bennett (1995) established four categories for describing or judging the merits of a figure built with interactive geometry software. A square, or other object, is either a Drawing, Underconstrained, Overconstrained, or Appropriately Constrained. Figure 7.1 shows a square that reveals itself to be a drawing when any vertex is dragged (see the sketch called “Draw vs. Construct.gsp” on the CD accompanying this yearbook). In this instance, the supposed square was created by eyeballing and then measuring the lengths and angles formed by four segments so they would appear equal in length and posi-
tioned at right angles. Without any geometric constraints built into the picture, any perturbation of the object deforms it into an arbitrary quadrilateral.

Figure 7.2 shows an underconstrained square. It has four built-in right angles but no constraints to keep its length equal. Dragging vertex $B$ deforms the square into a rectangle.

In contrast to an underconstrained square, an overconstrained square does retain four equal sides when dragged. These sides, however, remain steadfastly fixed in length and thus depict only a single square.

Finally, figure 7.3 shows a common method for building a bona fide square with appropriate constraints. The construction begins by drawing a circle with center $D$ and a point $A$ on its circumference. Point $A$, known as a "control point," changes the size of the circle when dragged, as does point $D$. By first drawing segment $DA$ and then using a perpendicular command, the user can construct a segment $DC$ of equal length. In a similar manner, segments $AB$ and $BC$ are constructed perpendicular to segments $DA$ and $DC$ respectively, guaranteeing that all four of the quadrilateral’s angles measure 90 degrees. With these construction features in place, $ABCD$ may grow, shrink, or rotate when tugged but will always remain square.

The remainder of this article highlights several interviewees' square-building efforts. Their methods expand on the construction distinctions above and raise new questions concerning the definition of an interactive geometry square.

**Norman's Square**

When Norman begins construction on his square, the interviewer tells him that he'll first need to "learn some things" about the software's menu items (such as the Perpendicular Line command). Undaunted, Norman assures him, "No, you can do it in a much easier way."

Norman draws a square $ABCD$ by freehand, carefully estimating the positions and lengths of the four on-screen segments. He then selects the software's circle tool and draws a circle that originates roughly in the center of $ABCD$ and extends out to its four vertices (see fig. 7.4 and the sketch called "Norman.gsp" on the CD accompanying this yearbook). Norman draws the circle so that point $A$ serves as its control point.

Norman drags point $A$ and admits that his square has some problems. Since its lengths and angles were estimated (or drawn) rather than constructed, $ABCD$ deforms into a random quadrilateral. Judged by Finzer and Bennett's classification, Norman has himself a drawing.

Yet beneath this seeming failure lies the germ of a good idea. When a circle expands, it retains its shape. If $ABCD$ can be linked to the circle, perhaps it, too,
will retain its shape. Because Norman drew his circle after forming \(ABCD\), none of the points \(B, C,\) and \(D\) remains attached to its circumference. Norman wants all four points of \(ABCD\) to move in unison with the circle, so he reverses course.

Norman begins afresh by drawing another circle. Only then does he draw \(ABCD\), placing each of its four vertices along the circle by visual inspection. Again, point \(A\) serves as the circle's control point. When Norman drags point \(A\) of his new square randomly across the screen, the behavior of \(ABCD\) exhibits more regularity than his previous attempt. Now, all four points of the quadrilateral remain attached to the circle when it moves. Aside from segments \(AB\) and \(AD\), the entire figure grows and shrinks proportionately. In other words, \(BC = CD\) and \(\angle BCD = 90\) degrees regardless of the quadrilateral's size (see fig. 7.5). Some careful dragging of point \(A\) yields more regularity still: if point \(A\) is dragged in a northwest direction (keeping the measure of \(\angle DAB\) equal to 90 degrees), \(ABCD\) remains a square while expanding.

**Commentary**

Norman’s square-building technique stays clear of the perpendicular and parallel line construction tools. Without the use of these menu items, it is hard to imagine how anything he builds could rise above the category of drawing. Yet through the clever use of a circle, he manages to build a quadrilateral \(ABCD\) that grows and shrinks, all the while approximating a square, provided point \(A\) moves in a controlled path.

Norman’s efforts fall short of an appropriately constrained object, but there is still much to admire in his work:

1. *Economy and speed.* Building figure 7.3's square demands multiple trips to the software's Perpendicular or Parallel Line commands. It also requires the user to hide construction lines and replace them by segments. By contrast, Norman's method involves no hidden lines and no menu selections. It can be completed in under a minute, qualifying it as a quick-and-simple means of illustrating a multitude of squares.

2. *Mathematical integrity.* None of the construction elements built into figure 7.1's drawing of a square help to keep its four sides equal in length. By contrast, Norman's method uses the symmetry of a circle to achieve four (sometimes) equal lengths.

Because Norman's construction exhibits "square" behavior, one might classify it an *underconstrained* square. This assessment, however, leaves room for disagreement. Norman drew his square by eyeballing the correct locations for its vertices. As such, the angle measures of \(ABCD\) were not precisely 90 degrees, and its lengths were not perfectly equal. These inaccuracies can be corrected by measuring and adjusting the particulars of \(ABCD\), but is measuring a legitimate part of the construction process? Traditionalists would likely say no.

Regardless of which side of this debate one chooses to accept, the message here is more general: deciding whether an object built with interactive geometry software is a square, contains some degree of "squareness," or is not a square can be surprisingly difficult and open to multiple perspectives.
David and Ben's Square

David and Ben begin their square construction by drawing a circle and placing points B and D on its circumference (see the sketch called David and Ben.gsp on the CD accompanying this yearbook). They add radii \( CB \) and \( CD \) to the picture and construct perpendicular lines \( j \) and \( k \) through points B and D, respectively (see fig. 7.6a).

David and Ben's construction is similar to the appropriately constrained square in figure 7.3, but with one difference: whereas \( \angle BAD \) in figure 7.3 always measures 90 degrees, \( \angle BAD \) in figure 7.6 can assume a variety of measurements, since points B and D move independently. Only when \( CB \) is perpendicular to \( CD \) is \( ABCD \) square (fig. 7.6b).

The interviewer questions David and Ben about \( ABCD \) but soon discovers that neither interviewee intends this quadrilateral to serve as his final square:

\[ \text{Int.}: \quad \text{You're trying to build a square. What are the features you're trying to build into it?} \]

\[ \text{Ben}: \quad \text{Equal sides and ninety-degree angles, all four.} \]

\[ \text{Int.}: \quad \text{Check this thing out by moving it in various ways. And find out whether any of the properties that you're looking for are} \]

Fig. 7.6. David and Ben's construction method

there. For example, you do have some ninety-degree angles that stay all the time.

\[ \text{Ben}: \quad \text{Well, we don't have a full square yet so we can't really tell.} \]

\[ \text{David}: \quad \text{Yeah we do. That's a full square [points to} ABCD].} \]

\[ \text{Ben}: \quad \text{Oh yeah. I wasn't looking at that part.} \]

\[ \text{David}: \quad \text{Actually, I wasn't really thinking of that either. I was thinking we'd go all the way around it [the circle].} \]

\[ \text{Ben}: \quad \text{Yeah, I was too.} \]

David and Ben plan to build a square that circumscribes (goes “all the way around”) their circle. They resume their construction by placing points E and F on the circle’s circumference, drawing CE and CF, and constructing lines perpendicular to the two segments (see fig. 7.7a).

By dragging points E and F to their proper locations, David and Ben are able to make their quadrilateral look like a square (see fig. 7.7b). Of course, any subsequent dragging of D, B, E, or F will deform the square back into an arbitrary quadrilateral.

At this point, the interviewer intervenes, explaining that since segments BF and DE were drawn (as opposed to constructed) perpendicular to each other, the illustration will never be an appropriately constrained square. Not to be deterred by this observation, David suggests a simple fix:

Fig. 7.7. David and Ben continue their square construction.
**David:** Wait, wait, wait. I have a question. If you hide points D, A, B, E, and F, and you could only move point C, would that work?

Sure enough, when David hides all points except the circle’s center (fig. 7.8a) and then drags point C, their quadrilateral grows and shrinks, always remaining a square regardless of point C’s location (fig. 7.8b). David and Ben give each other a congratulatory high-five and move on to the next challenge.

![Fig. 7.8. Hiding every point except C yields a resizable square.](image)

**Commentary**

As soon as David and Ben placed independent points B and D on their circle (fig. 7.6), I was tempted to curtail their work. My experience with appropriately constrained squares told me there was too much variability in their construction. David and Ben intended angle BCD to measure 90 degrees, but I knew a simple tug of either point B or D would alter the angle’s measure.

David and Ben’s solution to this problem—hiding nearly every point in their sketch (fig. 7.8)—struck me at first as cheating. Yes, dragging point C did enlarge and shrink their square perfectly well, but the hidden points B, D, E, and F could “mess up” the shape. Was it fair to remove these points from view, thus making them inaccessible to dragging by the mouse? Strictly speaking, yes. Hiding extraneous construction points is perfectly legal under the laws of interactive geometry constructions. Yet if David and Ben had constructed (as opposed to drawn) the segments perpendicular to each other, then there would be no need to hide points. Any point, visible or otherwise, would maintain the quadrilateral’s square shape when dragged.

**Conclusion**

In the language of interactive geometry, a square is created as either a drawing or a construction. To be more specific still, a square can be categorized as a drawing, underconstrained, overconstrained, or appropriately constrained. Theoretically, the differences among these four Finzer and Bennett (1995) categories create an unambiguous scheme for classifying students’ construction attempts. In practice, these distinctions proved decidedly murky. Students’ efforts at building geometric shapes from scratch sometimes defied categorization as “right” or “wrong.” The prevailing classification structure, although helpful, did not account for creative construction techniques.

Norman, David, and Ben built squares with minimal, if any, use of common interactive geometry software tools like “perpendicular line.” Through clever application of a circle’s symmetry, these students were able to build squares that, to varying degrees, maintained their “squareness” when dragged. Such nontraditional approaches serve as a challenge to teachers who assess the construction efforts of their students.

Students’ work may not fit strict construction standards, yet they may still contain nuggets of mathematical insight. If a student builds a square that sometimes, but not always, maintains its squareness (and is thus “underconstrained”), should she be commended for her work? I would argue yes. Although the quadrilateral lacks “appropriate constraints,” it is not small achievement to elevate a square above the category of “a drawing.” Teachers must look freshly at each of their students’ interactive geometry constructions and not adopt a lockstep appraisal method.

If several students in a class build squares using different techniques, a teacher can use this opportunity to ask them to compare and contrast the merits of each construction. What characteristics of each object qualify it as a square? Where does each square fall short? Can the class form a consensus on how an interactive geometry square should behave? Student-led critiques may be more effective than teacher-imposed definitions.
An Anthropological Account of the Emergence of Mathematical Proof and Related Processes in Technology-Based Environments

F. D. Rivera

Man’s use of mind is dependent upon his ability to develop and use “tools” or “instruments” or “technologies” that make it possible to express and amplify his powers…. It was not a large-brained hominid that developed the technical-social life of the human; rather it was the tool-using, cooperative pattern that gradually changed man’s morphology…. What evolved as a human nervous system was something, then, that required outside devices for expressing its potential.

—Jerome S. Bruner, Toward a Theory of Instruction

Current research on mathematical thinking and understanding has produced important information about how learners construct knowledge. However, the psychological orientation of many of these studies somehow downplays the instrumental role that tools play in cognitive processing despite the anthropological fact that all human experience is mediated by and structured through material, semiotic, and technological systems. In this paper, mathematical processes in technology-based environments refer to context-specific operations and actions that learners exhibit as they develop facility with different functions of technological tools that have been purposefully constructed to extend and amplify human capabilities. Thus, both epistemological and pragmatic concerns about the role of technological tools in the development of mathematical processes address the relationship between and among learner, tool, and task (see fig. 8.1), including the individual and social transformations that emerge from the interaction.

Editor’s note: The CD accompanying this yearbook contains six Cabri activities that are relevant to this article. For more information on Cabri, contact Cabri at www-Cabri.imag.fr/Cabri/index-e.html or at education.ti.com/us/product/software/cabri/features/features.html.