Cantor and Infinite Sets
Galileo and the Infinite

• There are many whole numbers that are not perfect squares: 2, 3, 5, 6, 7, 8, 10, 11, ... and so it would seem that “all numbers, including both squares and non-squares, are more than the squares alone.”

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, ...
Galileo and the Infinite

• “Not only so, but the proportionate number of squares diminishes as we pass to larger numbers, thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10000, we find only 1/100 part to be squares; and up to a million only 1/1000 part...”
Galileo and the Infinite

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Galileo and the Infinite

“If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.”

```
1  2  3  4  5  6  7  8
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1  4  9 16 25 36 49 64
```
Galileo and the Infinite

• But the number of roots must be the same as the number of whole numbers, because each whole number is the root of its corresponding perfect square.

• So there must be the same number of squares as there are numbers in general.

• What did Galileo conclude from this?
“So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes ‘equal,’ ‘greater,’ and ‘less,’ are not applicable to infinite, but only to finite, quantities.”
And....

• In many ways, that’s pretty much where it stood for 3 or 4 centuries, at least in Europe.

• Enter Georg Cantor.
Georg Cantor

- Born 1845 in St. Petersburg to a Danish father and a Russian mother.
- Moved to Germany as a teenager.
- Went to school in Germany, receiving a doctorate from University of Berlin in 1867.
Georg Cantor

• Did early work in number theory and analysis.
• Proved a theorem on the uniqueness of trigonometric series that was tackled unsuccessfully by many mathematicians, including Dirichlet and Riemann.
Georg Cantor

• Appointed to teach at Halle in 1869, and became friends with Richard Dedekind in 1872.
• Began his research into infinite sets.
• Suffered from bouts of depression beginning in May 1884, continuing off and on for the rest of his life.
Georg Cantor

- Experienced quite a bit of resistance to his ideas, in particular from Leopold Kronecker.
- He also had his supporters, among them Magnus Mittag-Leffler.
- For much of his later life, he wrestled with a mathematical problem called the *Continuum Hypothesis*.
- Died in 1918 in a Sanitorium.
Georg Cantor

• At one time it was thought that his struggles with mental illness were brought on by his mathematical struggles; it still makes a good story: mathematician struggles with ideas of infinity, can’t prove theorem, and goes insane.

• However, it is more likely that he suffered from bipolar disorder or clinical depression, and happened to be struggling with some serious mathematics at the same time.
Cantor’s Mathematics of the Infinite

• Implicit in Cantor’s early work is the idea of sets having the same number of elements if there is a one-to-one correspondence between their elements. We usually say that the sets have the same *cardinality*.

• A distinguishing feature of *finite* sets is that they cannot have the same cardinality as a proper subset.

• A distinguishing feature of *infinite* sets is that they *can* have the same cardinality as a proper subset.
Cantor’s Mathematics of the Infinite

• Thus there are as many even counting numbers as there are counting numbers, and also as many counting numbers as:
  – Odd counting numbers
  – Positive powers of 2
  – Perfect squares
  – Multiples of 17
  – Many, many other sets.
Cantor’s Mathematics of the Infinite

- Thus there are as many even counting numbers as there are counting numbers, and also as many counting numbers as:
  - Odd counting numbers
  - Positive powers of 2
  - Perfect squares
  - Multiples of 17
  - Many, many other sets.

- If we make this claim, we need to show the one-to-one correspondences. Can we name them?
Cantor’s Mathematics of the Infinite

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Cantor’s Mathematics of the Infinite

• Thus there are as many even counting numbers as there are counting numbers, and also as many counting numbers as:
  – Odd counting numbers \[ y = 2n - 1 \]
  – Positive powers of 2
  – Perfect squares
  – Multiples of 17
  – Many, many other sets.
Cantor’s Mathematics of the Infinite

• Thus there are as many even counting numbers as there are counting numbers, and also as many counting numbers as:
  – Odd counting numbers \( y = 2n - 1 \)
  – Positive powers of 2 \( y = 2^n \)
  – Perfect squares
  – Multiples of 17
  – Many, many other sets.
Cantor’s Mathematics of the Infinite

• Thus there are as many even counting numbers as there are counting numbers, and also as many counting numbers as:
  – Odd counting numbers \( y = 2n - 1 \)
  – Positive powers of 2 \( y = 2^n \)
  – Perfect squares \( y = n^2 \)
  – Multiples of 17
  – Many, many other sets.
Cantor’s Mathematics of the Infinite

• Thus there are as many even counting numbers as there are counting numbers, and also as many counting numbers as:
  – Odd counting numbers \( y = 2n - 1 \)
  – Positive powers of 2 \( y = 2^n \)
  – Perfect squares \( y = n^2 \)
  – Multiples of 17 \( y = 17n \)
  – Many, many other sets.
Cantor’s Mathematics of the Infinite

• Any infinite set with the same cardinality as the positive whole numbers is said to be countably infinite, or sometimes just countable.

• Among the countable sets are:
  – Integers
  – Negative integers
  – Even integers
  – Multiples of $n$ for any integer $n$. 
Countable Sets

• It is easy to believe that any infinite subset of a countable set is countable.

• What about these sets:
  – Fibonacci numbers
  – Primes
  – Composites
  – Integers greater than $10^{10^{10}}$
Cantor’s Mathematics of the Infinite

• So this leads to the question of whether Galileo was right – Is every infinite set countable? Is there just “infinite” and that’s all we can say?

• As a first guess, maybe the rational numbers form a bigger set. After all, between any two integers there is an infinite number of rationals, and between each of those rationals there is an infinite number of rationals, and between each of those rationals there is.....
Cantor’s Mathematics of the Infinite

• Cantor answered this question in 1873. He did this by showing a one-to-one correspondence between the rational numbers and the integers.

• Rational numbers are essentially pairs of integers – a numerator and a denominator. So he showed you can line up all pairs of integers to correspond with the integers. We illustrate this with positive integers, but including negatives is an easy extension.
### The Rationals Are Countable

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The Rationals are Countable

• Admittedly, the one-to-one correspondence in this case is kind of a mess. There are actually other theorems we’ll talk about later that make it easier to show two sets have the same infinite size.
Countable Sets

• With similar kinds of methods, you can show a lot of other infinite sets are countable:
  – Ordered triples, quadruples, etc. of integers.
  – Ordered pairs, triples, quadruples, etc. of rationals.

• Using other techniques, Cantor was able to show that the algebraic numbers, the numbers that are roots of polynomial equations with integer coefficients, were countable.
Is Everything Countable?

• So it seems reasonable to ask if there are any infinite sets that are not countable.
• Cantor was able to answer that one, too.
• He used what has come to be known as a diagonalization argument. It’s one of my favorites.
The Real Numbers are Not Countable

• Let’s prove the set of real numbers between 0 and 1 is not countable.
• Suppose otherwise; then there is a way to list them (i.e., a 1 − 1 correspondence between them and the positive integers):
The Reals are Not Countable

1. .012348737267398390200305377…..
2. .530000000000000000000000000000…..
3. .314159265494000000018475904…..
4. 1862824572038598702000000000…..
5. 67982543189375514614614614614…..
6. .33333333333333333333333333333333……
7. .873648950020330033000330003300…..
8. .10100100010000100000100000100000…..
9. .8765432187654321876543218765432187……
...

• So we have a list of all real numbers, written as decimal fractions, between 0 and 1. I’ve given an example of what the beginning of the list might look like.
The Reals are Not Countable

1. \(0\ldots\) 
2. \(0.5\ldots\) 
3. \(0.3\ldots\) 
4. \(0.186\ldots\) 
5. \(0.6798\ldots\) 
6. \(0.333\ldots\) 
7. \(0.8736\ldots\) 
8. \(0.101\ldots\) 
9. \(0.876\ldots\) 
...

- Now, in the first real number, mark the first digit after the decimal. In the second, mark the second digit after the decimal, and so on, in general marking the \(n\)th digit in the \(n\)th decimal in the list.
The Reals are Not Countable

1. 012348737267398390200305377.....
2. 53000000000000000000000000000.....
3. 314159265494000000018475904.....
4. 186282457203859870200000000.....
5. 67982543189375514614614614.....
6. 333333333333333333333333333333.....
7. 87364895002033000330003300.....
8. 101001000100010000010000010000.....
9. 87654321876543218765432187.....
...

- We are going to write a new decimal number by choosing each digit to be different from the digits we marked in the last step. So for my first digit, I choose something that isn’t a 0; for my second, something other than 3; next, something other than 4, and so on: 0.145118629...
The Reals are Not Countable

1. 012348737267398390200305377.....
2. 5300000000000000000000000000000.....
3. 314159265494000000018475904.....
4. 1862824572038598702000000000....
5. 679825431893755514614614614.....
6. 333333333333333333333333333.....
7. .8736489500203300033000330003300.....
8. .10100100010000100000100000.....
9. .8765432187654321876543218765432187.....
...

• I claim that the decimal fraction I created is nowhere in the list, since it is different from the nth decimal fraction in the list, precisely at the nth decimal place. So not all the decimals were in the list, contradictory to our claim.
The Reals are Not Countable

1. 012348737267398390200305377.....
2. 5300000000000000000000000000.....
3. 314159265494000000018475904.....
4. 1862824572038598702000000000....
5. 6798254318937551461461461461....
6. 3333333333333333333333333333.....
7. 873648950020330003300033003300.....
8. 10100100010000100000100000100000.....
9. 8765432187654321876543218765432187......
...

• This is Cantor’s famous “diagonalization” argument, which has become a standard tool in many branches of mathematical logic, including recursion theory and computability.
Countable and Uncountable

• So where are we?
• There are infinite sets that are countable, and infinite sets that are “bigger,” in particular the real numbers. Such sets are, believe it or not, called *uncountable*.
• Furthermore, since the rationals are countable, and the rationals and irrationals together make up the reals, we know that the irrational numbers are uncountable.
Other Sets of the Same Cardinality as the Reals

• The transcendental numbers – those that are not algebraic – form an uncountable set.
• So do all the points in the plane, and all the points in 3-space, and 4-space, etc. You might expect this since forming pairs and triples didn’t seem to make the infinite set “bigger” with the rationals.
• All real-valued \( n \) by \( m \) matrices form an uncountable set.
Other Sets of the Same Cardinality as the Reals

• Complex numbers
• All sequences of real numbers.
• All sequences of integers.
• The set of all \textit{continuous} functions from $\mathbb{R} \rightarrow \mathbb{R}$.
• The set of all \textit{finite} sequences of real numbers.
• The set of all subsets of integers.
So….Is There Anything Bigger?

• Glad you asked.
• Given any set $A$, the *Power Set* $\mathcal{P}(A)$ is the set of all subsets of $A$.
• Example: if $A = \{x, y, z\}$ then $\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$
• For finite sets, it’s clear that the power set is much larger than the set itself. In fact, if the set has $N$ elements, the power set has $2^N$. 
Time Out: Ordering Cardinalities

• We’ve talked about how to tell if two sets A and B are equal in size (there’s a one-to-one correspondence), and we know how to tell that they are not equal (no such correspondence exists).
• What about order? If we have = and ≠, can we get < or ≤?
• Of course, you know I wouldn’t ask if we couldn’t do it.
Time Out: Ordering Cardinalities

• Turns out the more natural idea (or at least the easier one) for infinite sets is $\leq$ rather than $<$.  
• We say $A \leq B$ if there is a one-to-one correspondence between $A$ and a subset of $B$.  
• Intuitively, this means $B$ has a subset the same size as $A$. So $B$ could be bigger, or as we have seen, it could be the same size. Hence, $\leq$.  

Time Out: Ordering Cardinalities

• So the way we usually show that $A < B$ is to show $A \leq B$ and $A \neq B$. Or, that there is a one-to-one correspondence between $A$ and a subset of $B$, but not between $A$ and all of $B$.
• This is pretty much what we did to show that the reals were strictly bigger than the whole numbers. What was the one-to-one correspondence with a subset of the reals?
Time Out: Ordering Cardinalities

• So one last question. For a finite set like $A = \{x, y, z\}$ and its power set $\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$, what would you use for the one-to-one correspondence from $A$ to a subset of $\mathcal{P}(A)$?

• It’s pretty clear that no one-to-one correspondence could exist between a finite set and its full power set.
Cardinality of Power Sets

- Cantor succeeded in proving that for all sets, even infinite sets, the cardinality of the power set is strictly larger than that of the set.
- (In fact, the cardinality of the Reals is exactly the cardinality of the power set of the Integers.)
- So the power set of the Reals is larger than the reals. And the power set of the power set of reals is larger still. And so on, forever.
Infinite Cardinalities

- So there are infinite towers of larger and larger infinite sets.
- Thus, there is not just one size of infinity, or even two, but infinitely many. And, there are ways of getting larger cardinals than those obtained by power sets.
- A brave new infinite world.
The Continuum Hypothesis

• So the “smallest” infinite sets we know of are the countably infinite sets -- the counting numbers, or integers, or rationals, etc.

• We know that the real numbers, which are the same size as the power set of integers, form a larger set.

• Question: is there an infinite set larger than the integers, but smaller than the reals? Is there something “in between”? 
The Continuum Hypothesis

• The statement that there is no such set is known as the Continuum Hypothesis.
• Cantor tried to prove it, and often thought he had a proof, only to realize that he didn’t. At times, he thought he had proved it wasn’t true, again only to realize that the proof was flawed. He spent the last part of his life working on this problem, and it probably didn’t help his depression any.
Aside:

• Does Cantor’s experience with the CH remind you of any other similar struggles we’ve talked about?

• In particular, the trying to prove it / trying to disprove it cycle?

• And not being successful?

• Anyone? Anyone? Beuller?
The Continuum Hypothesis

- As it turned out, it was shown in the early 1900’s that the Continuum Hypothesis was independent of the axioms of set theory. That is, there is a model of set theory where it is true, and a model where it is false. We’ll tell the story of this remarkable discovery later in class.

- In fact, we’ll let Dr. du Sautoy tell some of it.
Cardinal Arithmetic

• Cantor also extended arithmetic to include infinite as well as finite cardinalities. The basic laws of arithmetic for infinite cardinalities are somewhat strange. For example, to add two countable sets together produces just another countable set. We might write this as:

\[ \aleph_0 + \aleph_0 = \aleph_0. \]
Ordinals vs Cardinals

• We have (at least) two ideas associated with number: *how many* – the idea of cardinality – and *order* – the idea of ordinality. They are exemplified by the difference between
  • One, two, three, four, five, six, seven,….
  • First, second, third, fourth, fifth, sixth, seventh, …
Ordinals vs Cardinals

• Cantor also worked with infinite ordinals. The first infinite ordinal and the first infinite cardinal are the same:

• $\omega = \{1, 2, 3, 4, 5, 6, \cdots\} = \aleph_0$

• But notice that we can define a “next” ordinal by adding an element that is larger than all the natural numbers. We call this set $\omega + 1$. It has the same cardinality, but different order properties, than $\omega$. 
Ordinals vs Cardinals

- We could keep this up, adding element after element, infinitely many times. We would then get:

- $\omega 2 = \{1, 2, 3, \cdots, \omega + 1, \omega + 2, \omega + 3, \cdots \}$
- This is two copies of $\omega$, one right after the other. It still has the same cardinality.

- $\omega 2 = \{1, 2, 3, 4, 5, 6, \ldots, 1, 2, 3, 4, 5, 6, \ldots \}$. Here, all the red numbers are bigger than all the black numbers, and ordering within the reds and blacks looks like the usual ordering of natural numbers.
- We could also form the set made of countably many copies of $2$:

- $2\omega = \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \ldots \}$
- Here, all the red numbers are larger than the corresponding black numbers, and ordering otherwise looks like ordering of natural numbers. It’s still countable, but has different order properties (e.g. each number has only finitely many before it).
Ordinals vs Cardinals

• We could pile up $\omega$ copies of $\omega$, one after the other, which would be $\omega^2$. It’s still countable.
• Of course, we could get $\omega^2 + \omega^3 + 7$, and a lot of similar things, before we got to $\omega^3, \omega^4$, and eventually, of course, $\omega^\omega$, and $\omega^\omega, \omega^\omega\omega, \omega^\omega\omega\omega$, and so on, until you get to $\varepsilon_0 = \omega^\omega^\omega^\omega\omega\ldots$, which is still countable.
Ordinals vs Cardinals

• And then of course there’s $\varepsilon_0 + 1, \ldots, \varepsilon_0 + \omega$, and so on, until you just want to crawl in a hole and die.
• Or else you’re like me and think, “Cool.”
• There are also towers of cardinals like this: small cardinals like you can get from power sets, inaccessible cardinals, Mahlo cardinals, reflecting cardinals, compact cardinals, unfoldable cardinals, ethereal cardinals, ineffable cardinals, remarkable cardinals, Ramsey cardinals,........
Summing Up

• David Hilbert described Cantor's work as: 
  ...the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity.

• I gotta say, I agree.