Saccheri Quadrilaterals

**Definition:** Let $\overline{AB}$ be any line segment, and erect two perpendiculars at the endpoints A and B. Mark off points C and D on these perpendiculars so that C and D lie on the same side of the line $\overline{AB}$, and $BC = AD$. Join C and D. The resulting quadrilateral is a **Saccheri Quadrilateral**. Side $\overline{AB}$ is called the **base**, $\overline{BC}$ and $\overline{AD}$ the **legs**, and side $\overline{DC}$ the **summit**. The angles at C and D are called the **summit angles**.

![Saccheri Quadrilateral Diagram](attachment: Saccheri_Quadrilateral.png)

**Lemma:** A Saccheri Quadrilateral is convex.

☐ By construction, D and C are on the same side of the line $\overline{AB}$, and by PSP, $\overline{CD}$ will be as well. If $\overline{AB}$ intersected $\overline{CD}$ at a point F, one of the triangles $\triangle ADF$ or $\triangle BFC$ would have two angles of at least measure 90, a contradiction (one of the linear pair of angles at F must be obtuse or right). Finally, if $\overline{AD}$ and $\overline{BC}$ met at a point E then $\triangle ABE$ would be a triangle with two right angles. Thus $\overline{BC}$ and $\overline{AD}$ must lie entirely on one side of each other. So, $\square ABCD$ is convex.
Theorem: The summit angles of a Saccheri Quadrilateral are congruent.

☐ The SASAS version of using SAS to prove the base angles of an isosceles triangle are congruent. □DABC ≅ □CBAD, by SASAS, so ∠D = ∠C. ■

Corollaries:

• The diagonals of a Saccheri Quadrilateral are congruent. (Proof: ∆ABC ≅ ∆BAD by SAS; CPCF gives AC = BD.)

• The line joining the midpoints of the base and summit of a quadrilateral is the perpendicular bisector of both the base and summit. (Proof: Let N and M be the midpoints of summit and base, respectively. Use SASAS on □NDAM and □NCBM. The angles at M and N are congruent by CPCF, and form a linear pair, so must have measure 90.)

• If each of the summit angles of a Saccheri Quadrilateral is a right angle, the quadrilateral is a rectangle, and the summit is congruent to the base. (Proof: Consider diagonal AC. HL gives ∆ADC ≅ ∆CBA so DC = AB.)
Lemma: If \( \square ABCD \) has right angles at \( A \) and \( B \), then: \( AD < BC \) iff \( \mu \angle C < \mu \angle D \), \( AD = BC \) iff \( \mu \angle C = \mu \angle D \), and \( AD > BC \) iff \( \mu \angle C > \mu \angle D \).

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\mu \angle C = \mu \angle D, \text{ and } AD > BC \text{ iff } \mu \angle C > \mu \angle D.
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\[\square ABCD\] is Saccheri and we have already shown \( \mu \angle C = \mu \angle D \). If \( AD < BC \), then let \( E \) be a point such that \( B*E*C \) and \( BE = AD \). Then \( \square ABED \) is Saccheri and \( \mu \angle ADE = \mu \angle BED \). \( E \) is interior to \( \angle ADC \), so \( \mu \angle ADC > \mu \angle ADE = \mu \angle BED \). But \( \mu \angle BED > \mu \angle BCD \) by exterior angle inequality, so \( \mu \angle ADC > \mu \angle BCD \). An exactly analogous proof will establish \( AD > BC \) implies \( \mu \angle C > \mu \angle D \).

We now have one direction of each “iff” statement. Now suppose \( \mu \angle C = \mu \angle D \). If \( AD \neq BC \), then \( AD < BC \) or \( AD > BC \). But we have shown this either gives \( \mu \angle C > \mu \angle D \) or \( \mu \angle C < \mu \angle D \), in contradiction to \( \mu \angle C = \mu \angle D \). So \( \mu \angle C = \mu \angle D \) implies \( AD = BC \). Similarly, if \( \mu \angle C > \mu \angle D \) but \( AD > BC \), then \( AD < BC \), and either \( \mu \angle C = \mu \angle D \) or \( \mu \angle C < \mu \angle D \), again contradicting \( \mu \angle C > \mu \angle D \). Similarly, \( \mu \angle C < \mu \angle D \) must imply \( AD < BC \).

(Note how we got the converse of each implication since the three implications taken together exhaust all the possibilities.)
**Theorem:** If the summit angles of a Saccheri Quadrilateral are acute, the summit has greater length than the base.

Given \( \Box ABCD \), connect the midpoints N and M of summit and base. Consider \( \Box MNBC \), with right angles at M and N (we proved this above). By the previous lemma, if \( \angle C \) is acute, and therefore less than \( \angle B \) in measure, \( MC > NB \). Similarly, considering \( \Box MNAD \), if \( \angle D \) is acute and therefore less than \( \angle A \) in measure, \( DM > AN \). Combining these, since \( D*M*C \) and \( A*N*B \), \( DC > AB \). ■

**Note:** If the summit angles are obtuse, we can just as easily, and in the exact same way, prove that the base is longer than the summit. Since we already know that if the summit angles are right, we have a rectangle, with summit and base of equal length, we can summarize in the following way:

If the summit angles of a Saccheri Quadrilateral are:

- acute, the summit is longer than the base
- right, the summit is equal to the base and the quadrilateral is a rectangle
- obtuse, the summit is shorter than the base.
In absolute geometry, we cannot establish that the summit angles are in fact right angles. But we can eliminate the “obtuse” case, and in the process, get another “almost Euclidean” result.

**Theorem:** The summit angles of a Saccheri Quadrilateral are either acute or right.

**Proof:** Begin with Saccheri Quadrilateral $ABCD$, with right angles at $A$ and $B$ (so below it is drawn “upside down” relative to how we have usually drawn it). Extend $BC$ to a point $E$ with $CB = BE$. Let $M$ be the midpoint of $AB$. Consider $\triangle EBM$ and $\triangle DAM$. Since $EB = CB = AD$, and $AM = BM$, we have $\triangle EBM \cong \triangle DAM$ by SAS. This means $\angle AMD = \angle BME$ by CPCF, and so $D*M*E$ (we have previously proved this). Thus we have a triangle $\triangle DEC$, and $\angle DEC \cong \angle ADE$.

Now the sum of the summit angles of $ABCD$ is $\mu \angle ADE + \mu \angle EDC + \mu \angle C$. The sum of the angles of triangle $\triangle DEC$ is $\mu \angle DEC + \angle EDC + \mu \angle C$. But these sums are equal, since $\angle DEC = \angle ADE$. Thus, since the angle sum of the triangle is no more than 180, so is the sum of the summit angles. Since the summit angles are equal in measure, they must be acute.

**Corollary:** The length of the summit of a Saccheri Quadrilateral is greater than or equal to the length of the base.
This lets us now prove one more interesting thing about Saccheri Quadrilaterals: They are, in fact, parallelograms.

**Definition:** A **parallelogram** is a convex quadrilateral in which opposite sides are parallel.

**Theorem:** A Saccheri Quadrilateral is a parallelogram.

Outline of proof: We have already noted that lines \( AD \) and \( BC \) cannot meet (if they met at a point \( F \), then \( \triangle ABF \) would have two right angles). Suppose \( CD \) and \( AB \) were to meet at a point \( G \). Either \( DCG \) or \( GDC \). WLOG suppose \( DCG \). Then since the summit angle at \( C \) is acute, the angle \( \angle GCB \), being supplementary, is obtuse. Similarly, the angle \( \angle GBC \) is supplementary to the right angle at \( B \). Thus \( \triangle GCB \) has two right or obtuse angles, a contradiction. The case for \( GDC \) is similar.
Theorem: The line joining the midpoints of two sides of a triangle has length less than or equal to one-half of the third side. (Note: in Euclidean geometry, the inequality is replaced by equality.)

Outline of proof: Let \( \triangle ABC \) be given, and locate midpoints \( M \) and \( N \) of \( \overline{AB} \) and \( \overline{AC} \), respectively. Drop perpendiculars from \( B \) and \( C \) to line \( \overline{MN} \) at \( B' \) and \( C' \), respectively. Since \( B \) and \( C \) lie on opposite sides of \( \overline{MN} \) from \( A \), they lie on the same side of \( \overline{MN} \) and just as for Saccheri Quadrilaterals we can show \( \square BCB'C' \) is convex.
Our next goal is to show that $\square BCB'C$ is in fact a Saccheri Quadrilateral. To do so, we drop a perpendicular from $A$ to $MN$ at point $Q$. There are three cases: (1) $\angle AMN$ and $\angle ANM$ are both acute; (2) one of $\angle AMN$ and $\angle ANM$ is right; and (3) one of $\angle AMN$ and $\angle ANM$ is obtuse: In Case 1, $B'MQNC'$. In Case 2, $B'=M=Q$, and $MNC'$. In Case 3, $Q'M'B'C'$ and $MNC'$. 

Case 1

Case 2

Case 3
We’ll do Case I here, and leave the others as exercises.

Now $AM = MB$ and $AN = NC$ by construction. Since they are vertical angles, $\angle AMQ \cong \angle BMB'$ and $\angle ANQ \cong \angle CNC'$. By HA, $\triangle AMQ \cong \triangle MB'Q$ and $\triangle ANQ \cong \triangle NCQ$. So $BB'\cong AQ \cong CC'$ and $\triangle BCB'C'$ is a Saccheri Quadrilateral. Now $B'M = MQ$, and $C'N = NQ$. Since $B'MQN\cong B'C'NQ$, $B'C' = B'M + MQ + QN + NC' = 2MQ + 2NQ = 2(MQ + NQ) = 2MN$, so $MN = \frac{1}{2}(B'C')$.

Since $B'C'$ is the base and $BC$ the summit of Saccheri Quadrilateral $\square BCB'C$, $B'C' \leq BC$. Thus, $MN = \frac{1}{2}(B'C') \leq \frac{1}{2}(BC)$. 