Homework 6 Answers

1. Using the general formula for a coordinate function for lines using the Euclidean metric (we developed this in class), I got \( f(x, y) = x\sqrt{1 + (-2)^2} = x\sqrt{5} \). I know that this is one possible coordinate function for the line, but it may not be the one that sends (0,3) to 0. So I plug in (0,3) into the function. If I don’t get 0, I may have to add a constant to the function to shift the “ruler” so that (0,3) corresponds to 0. But \( f(0,3) = 0 \cdot \sqrt{5} = 0 \), so I don’t need to add a constant.

2. I need another coordinate function that will reverse the direction of my ruler. I’m going to call this function \( g \), and my guess is that \( g(x, y) = -f(x, y) \), or in other words, my new function should just be the old function with a sign change. To check this out, I’m going to try plugging in a generic point on the line, \((x, y)\). Then \( g(x, y) = -f(x, y) = -x\sqrt{5} \), and this is the same real number \((x, y)\) originally got sent to except with a sign change. So I’ve reversed the direction of my ruler.

3. My old function \( f \) maps \((4, -5)\) onto the number \( 4\sqrt{5} \). I want it to map onto \(-7\). So I will create a new function \( h \) that adds \(-7 - 4\sqrt{5}\) to \( f \), or in other words, \( h(x, y) = x\sqrt{5} - 7 - 4\sqrt{5} \). Then \( h(4, -5) = 4\sqrt{5} - 7 - 4\sqrt{5} = -7 \), which is just what I wanted. Furthermore, this is merely a shift of the ruler, and so distances are still preserved, which means it will still work as a coordinate function for the line \( y = -2x + 3 \).

4. a. My new function \( g(x, y) = x\sqrt{5} + r \) will be a coordinate function for \( y = -2x + 3 \). I know it is one-to-one, because it is linear. Now I just need to make sure that it preserves distance. I do this by showing that \( d(P, Q) = |g(P) - g(Q)| \). I’ll let my two points be \( P = (x_1, -2x_1 + 3) \) and \( Q = (x_2, -2x_2 + 3) \). Then
   \[
   d(P, Q) = \sqrt{(x_2 - x_1)^2 + [2x_2 - 3 - (2x_1 + 3)]^2} = \sqrt{(x_2 - x_1)^2 + 4(x_2 - x_1)^2} = \sqrt{5}|x_2 - x_1|, \text{ and}
   \[
   |g(P) - g(Q)| = \sqrt{5} + r - (\sqrt{5} + r) = \sqrt{5}|x_1 - x_2|. \text{ These are equal, so } g(x, y) \text{ is a coordinate function for } y = -2x + 3.
   
   b. This new function shifts the ruler 2 in the negative direction, because before, \((0, 3)\) mapped onto the number 0, and now it maps onto the number 2.

5. Let \( P = (x_1, -2x_1 + 3) \) and \( Q = (x_2, -2x_2 + 3) \). The distance between these points using the Taxicab metric is \( \rho(P, Q) = |x_2 - x_1| + |2x_2 + 3 - (-2x_1 + 3)| = |x_2 - x_1| + 2|x_2 - x_1| = 3|x_2 - x_1| \).
   
   Now if I let \( f(x, y) = 3x \), then \( |f(P) - f(Q)| = |3x_1 - 3x_2| = 3|x_1 - x_2| \), which is equal to the distance between \( P \) and \( Q \) in the Taxicab metric. And \( f(0, 3) = 0 \), so I don’t need to shift \( f \). So the answer is \( f(x, y) = 3x \).

6. a. So \( g(x) = -3x + 6 \). This function is linear, so it is one-to-one. Now I need to show that \( \rho(P, Q) = |g(P) - g(Q)| \). Let \( P = (x_1, -2x_1 + 3) \) and \( Q = (x_2, -2x_2 + 3) \). As we saw above,
\[ \rho(P, Q) = |x_2 - x_1| + 2x_2 + 6 - (2x_1 + 6) = 3|x_2 - x_1| \]

and
\[ |g(P) - g(Q)| = -3x_1 + 6 - (-3x_2 + 6) = 3|x_1 - x_2|, \]

so \( g \) is a coordinate function for the line.

b. The new function \( g \) reverses the direction of the ruler and maps \((0, 3)\) to 6, so it shifts the ruler 6 in the negative direction.

7. Let \( P = (x_1, -2x_1 + 3) \) and \( Q = (x_2, -2x_2 + 3) \). The distance between these points using the square metric is
\[ D(P, Q) = \max \{ |x_2 - x_1|, -2x_2 + 3 - (-2x_1 + 3) \} = \max \{ |x_2 - x_1|, 2|x_2 - x_1| \} = 2|x_2 - x_1|. \]

If we pick the function \( f(x, y) = 2x \), then \( |f(P) - f(Q)| = 2|x_1 - x_2| \). Since \( D(P, Q) = |f(P) - f(Q)| \), \( f \) is a coordinate function for \( y = -2x + 3 \) using a square metric.

8. Let \( P = (x_1, mx_1 + b) \) and \( Q = (x_2, mx_2 + b) \). The distance between these points using the square metric is
\[ D(P, Q) = \max \{ |x_2 - x_1|, |mx_2 + b - (mx_1 + b)| \} = \max \{ |x_2 - x_1|, m|x_2 - x_1| \}. \]

So
\[ D(P, Q) = m|x_2 - x_1| \]

if \( |m| \geq 1 \) and \( |x_2 - x_1| \) if \( |m| < 1 \). If we pick the function \( f \) such that
\[ f(x, y) = \begin{cases} mx, & \text{if } |m| \geq 1 \\ x, & \text{if } |m| < 1 \end{cases} \]

then \( |f(P) - f(Q)| = D(P, Q) \), which implies that \( f \) is a coordinate function for \( y = mx + b \) using a square metric.

9. If \( f(x, y) \) is a coordinate function for a particular metric given any two points \( P, Q \) on the line, \( PQ = |f(P) - f(Q)| \) (remember that \( PQ \) means the distance between \( P \) and \( Q \)). For any real number \( r \), the function \( g(P) = f(P) + r \) is also a coordinate function for the following reasons. First, if \( f \) is one-to-one, then so is \( g \), since adding \( r \) is just a shift of the ruler. Furthermore,
\[ |g(P) - g(Q)| = |(f(P) + r) - (f(Q) + r)| = |f(P) - f(Q)| = PQ, \]

so \( g \) is also a coordinate function.