Directions: In class, we took the Plane Separation Postulate as an axiom, and from it proved the Postulate of Pasch or as the book calls it, Pasch’s Axiom (Theorem 5.5.10, p.67). The idea of this problem is to take the Postulate of Pasch as a postulate and use it to prove the Plane Separation Postulate as a theorem. I will walk you through an outline of the proof, and you fill in the details.

Given (The Postulate of Pasch): If A, B, and C are any three distinct noncollinear points in a plane, and \( l \) is any line which also lies in that plane and passes through an interior point D of segment AB but not though A, B, or C, then \( l \) meets either AC at some interior point E, or BC at some interior point F, the cases being mutually exclusive.

Prove: If \( l \) is any line lying in any plane \( P \). The set of all points in \( P \) not on \( l \) consists of the union of two subsets \( H_1 \) and \( H_2 \) of \( P \) such that

1. \( H_1 \) and \( H_2 \) are convex sets.
2. \( H_1 \) and \( H_2 \) have no points in common.
3. If \( P \) lies in \( H_1 \) and \( Q \) lies in \( H_2 \), line \( l \) intersects the segment PQ.

The Plan: First we prove that, given any an arbitrary line \( l \), \( H_1 \) and \( H_2 \) exist; in fact, we will construct these two sets in a particular way. Then, we need to show that they have the properties listed above.

Step 1: Define \( H_1 \) and \( H_2 \) as follows: Let \( l \) be any line, and \( L \) a point on \( l \). Find points A and B with A - L - B (How?). Let \( H_1 \) be the set of points consisting of point A and all points \( P \) such that \( l \) does not intersect \( \overline{AP} \). Define \( H_2 \) similarly using B. It is crucial that you understand this definition, and what it means for a point to be in \( H_1 \). A point \( P \) is in \( H_1 \) if and only if line \( l \) does not intersect \( \overline{AP} \). In Figure 1, points D, E, F, G, H, I and J are all in \( H_1 \). Make sure you understand how a point gets to be in \( H_1 \) or \( H_2 \) before you go on.
Step 2: Show $H_1$ is convex. Recall that a set is convex if you can choose any two points in the set and show that the segment connecting them lies entirely in the set. So, we choose two points $P$ and $R$ in $H_1$, and show that any point $S$ on the segment connecting them is also in $H_1$. But, because $P$ and $R$ can be any two points, we need to consider 3 cases:

1. One of the two points is $A$; say $P = A$, and $A-S-R$. (See Figure 2)
2. $P$, $A$, and $R$ are collinear. (See Figure 3, which shows the case $P-A-R$) There are cases $P-A-R$, $A-R-P$, . . . How many of these do you need to consider? Be sure to consider all the ways point $S$ can be on the segment $PR$.
3. $P$, $R$, and $A$ are non-collinear, and suppose that $P - S - R$ (see Figure 4). In this case, show that $S$ is in $H_1$, by applying Pasch to $\triangle APR$, and then to $\triangle APS$. (You may note that showing $H_2$ convex has the same basic form.)
Step 3: Prove that every point off $l$ is in $H_1$ or $H_2$. (Pick an arbitrary point $T$ off $l$, and consider three cases: Point $T$ is in the interior of ray $\overline{LA}$ (see Figure 5); point $T$ is in the interior of ray $\overline{LB}$; point $T$ is off the line $\overline{AB}$ (see Figure 6).
Step 4: Prove that if P is in $H_1$ and Q is in $H_2$, with P and Q distinct, then l intersects $PQ$. Consider cases:

1. P is in ray $LA$ and Q is in ray $LB$;
2. P is in $LA$ and Q is off $LB$, and consider Pasch applied to $\triangle BPQ$
3. P is off $LA$ and Q is on $LB$ and consider Pasch applied to $\triangle APQ$;
4. P and Q are both off $AB$, with Pasch applied to $\triangle ABP$ and then to $\triangle BPQ$. (See Figure 7)

Figure 7

Step 5: Finally, show that $H_1$ and $H_2$ are disjoint, by supposing there is a point that is in each of $H_1$ and $H_2$ and reaching a contradiction.