Plane Separation, Angle Interiors, and Pasch

**Definition:** A set $S$ of points is said to be *convex* provided that whenever two points $A$ and $B$ are in $S$, the entire segment $AB$ is in $S$.

It is sometimes useful to know, and very easy to prove, that the intersection of two convex sets is convex.

**Axiom: The Plane Separation Postulate (PSP):**

If $l$ is any line then the set of all points not on $l$ consists of the union of two sets $H_1$ and $H_2$ such that:

1. $H_1$ and $H_2$ are convex sets;
2. $H_1$ and $H_2$ are disjoint sets; and
3. If point $A$ lies in $H_1$ and point $B$ lies in $H_2$, then the line $l$ intersects the segment $AB$.

**Notation:** We call $H_1$ and $H_2$ the two *sides* of $l$ or the two *half-planes* determined by $l$. If point $A$ is in $H_1$ and $B$ is in $H_2$, we will use the notion $H_A$ to mean the half-plane determined by $l$ and containing the point $A$, and say it is the *$A$-side of $l$*. Thus we can say $H_1 = H_A$, and $H_2 = H_B$. A point in $H_1$ and a point in $H_2$ are said to be on *opposite sides* of line $l$, whereas all points in $H_1$ are said to be on the *same side* of line $l$.

This axiom tells us that every line separates the plane into two half-planes, and that if you want to connect points in different half planes, the segment that connects them must intersect the
original line. We can now talk about “sides” of a line, i.e. the halfplanes determined by that line.

**Definition:** Two rays \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) having the same endpoint are **opposite rays** if the two rays are unequal but \( \overrightarrow{AB} = \overrightarrow{AC} \). Otherwise they are **nonopposite**.

**Note** that \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) are opposite provided \( B \neq A \neq C \).

**Definition:** An **angle** is the union of two nonopposite rays \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) sharing the same endpoint. We denote the angle by \( \angle BAC \) or \( \angle CAB \). The point A is called the **vertex** of the angle, and the rays \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) are called its **sides**.

**Note** that in our geometry there are no “straight” angles.

**Definition:** Given an angle \( \angle ABC \), the **interior** of \( \angle ABC \) is defined as follows: If \( \overrightarrow{BA} \neq \overrightarrow{BC} \), the interior is the set of all points that simultaneously lie on the A-side of \( \overrightarrow{BC} \) and the C-side of \( \overrightarrow{BA} \). If \( \overrightarrow{BA} = \overrightarrow{BC} \), the interior is the empty set.

**Note** that the interior of an angle, being the intersection of convex sets, is also convex.

**Definition** (Just to make it officially part of our development of neutral geometry): Three points A, B, and C are said to be **collinear** provided they lie together on the same line. Otherwise, they are said to be **noncollinear**.
Note that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite provided they are unequal and $A$, $B$, and $C$ are collinear. If $A$, $B$, and $C$ are noncollinear, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ must be nonopposite.

**Definition:** A triangle is the union of three segments (called its sides), whose endpoints (called its vertices) are taken, in pairs, from a set of three noncollinear points. Thus, if the vertices of a triangle are $A$, $B$ and $C$, then its sides are $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$, and the triangle is then the set defined by $\overrightarrow{AB} \cup \overrightarrow{BC} \cup \overrightarrow{AC}$, denoted by $\triangle ABC$. The angles of $\triangle ABC$ are $\angle A = \angle BAC$, $\angle B = \angle ABC$, and $\angle C = \angle ACB$.

**Theorem (Postulate of Pasch):** Suppose $\triangle ABC$ is a triangle, and $l$ is any line that does not pass through a vertex of $\triangle ABC$ but passes through an interior point $D$ of one of its sides. Then $l$ meets exactly one of the other two sides at an interior point.

**Note:** This is called the *Postulate of Pasch* because Moritz Pasch (1843-1930) first recognized its importance, and also because it can be taken as an axiom (or postulate) and PSP can then be proved from it. Thus, it is equivalent to PSP. Interested students can consult pages 132-133 of George E. Martin’s *The Foundations of Geometry and the Non-Euclidean Plane* (Springer Verlag, 1975) for a proof of this fact. Or, you can work through a worksheet I adapted from that source, that provides a little more detail and guidance, and prove it yourself! See “Extras” on the course Web page.
The textbook states the theorem for the specific case of passing through $\overline{AB}$, and we can assume that is the case without loss of generality (WLOG).

Let $A$, $B$, $C$ be distinct noncollinear points in a plane, and $l$ be any line in that plane passing through an interior point $D$ of segment $\overline{AB}$, but not through point $C$. Then $A$-$D$-$B$, and WLOG $A \in H_1$ and $B \in H_2$. Since $l$ doesn't pass through $C$, $C$ is either in $H_1$ or $H_2$. Consider each case.

Case 1: $C$ is in $H_1$. Then since $B \in H_2 \overline{CB}$, must intersect $l$ by PSP (the intersection point is interior to $\overline{CB}$ since neither $C$ nor $B$ are on $l$). Moreover, since $A \in H_1$ and $H_1$ is convex, segment $\overline{AC}$ lies in $H_1$ so cannot intersect $l$. 

![Diagram of points A, B, C, D, and line l with regions $H_1$ and $H_2$]
Case 2: C is in $H_2$. Same stuff, different names.

Thus, either $l$ meets either $\overline{CB}$ at some interior point, or $\overline{AC}$ at some interior point. The cases are mutually exclusive since Case 1 and Case 2 are.