Looking at the complexity of two young children’s understanding of number

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Abstract

This paper presents a qualitative study that investigated two third-grade students’ understanding of number. The children were videotaped while they worked to record everything they knew about the number, 72. Their artifacts and conversations were then analyzed using the Pirie–Kieren dynamical theory for the growth of mathematical understanding as the theoretical framework. The results of the video analysis revealed the two students’ understanding of natural numbers as being conceptually complex and existing in several different realms of the Pirie–Kieren model. Significant instances of \textit{Primitive Knowing}, \textit{Image Making}, \textit{Image Having}, \textit{Property Noticing}, \textit{Formalizing}, \textit{Observing}, as well as how the children’s understanding existed beyond “Don’t Need” Boundaries are identified and examined in detail. Other features of the model — \textit{Structuring}, \textit{Inventising}, \textit{Folding Back}, and \textit{Connected Understanding} — are also explained and possible examples illustrating these kinds of mathematical thinking in relation to the two children’s understanding of number are offered.

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1. Introduction

What do you get when you ask two young children to work as partners and address the following question, “What do you know about 72?” As part of a project in which we offered children “answers” and encouraged them to create “questions,” we gave each pair of children in a second- and third-grade class a “secret number” (i.e., an answer) and asked them to write down everything they could say about it as a set of “clues” (i.e., questions). The set of clues were then to be given to another pair of students the following day so that they could solve for and identify the secret number. We wanted to create an inviting space in which we could assess the children’s mathematical understanding rather than merely their ability to produce answers to standard arithmetic questions. In this paper, we share “what we got.”

We videotaped two third-grade children (i.e., Sam and Sammy)\textsuperscript{1} who were given the number, 72 and then analyzed the session for what it revealed of their understanding of number. In assessing the two children’s mathematical
understanding, it was not our intention to measure it either quantitatively or in terms of static levels. Rather, we set out to examine the breadth and depth of the children’s understanding, to investigate the complexity of their concept of number, and to observe the facility with which they integrated, distinguished, and accessed their informal and formal mathematical realms of knowing. For this reason, we chose to use the Pirie–Kieren dynamical theory for the growth of mathematical understanding as our theoretical framework. Our discussion focuses on explaining and demonstrating how the Pirie–Kieren theory was used to reveal the mathematical knowings embedded in the two children’s written records and conversations about the number, 72.

2. The Pirie–Kieren theory

This theory takes understanding to be a dynamical, leveled but non-linear, transcendentally recursive process of reorganizing one’s knowledge structures. It is a theory for growth of understanding of a specified topic by a specific person or group. Thus, it serves as a theoretical lens through which an individual’s or group’s mathematical understanding can be illuminated, described, and assessed as it unfolds or, in the forms in which it exists. Mathematical understanding is viewed as being brought forth by interrelated and fluid processes, evolving in a fractal-like manner from existing knowing actions (e.g., Kieren, 1990; Pirie and Kieren, 1989; Pirie & Kieren, 1989, 1994b).

It is important to note that Pirie and Kieren do not characterize the structure of their theory as being hierarchical or linear, although the two-dimensional model (see Fig. 1) may give this impression. The realms of mathematical knowings within this model exist as embedded, unbounded circles that are recursive and self-similar to one another. As such, Pirie and Kieren (1989) emphasize that “each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further is constrained by those without” (p. 8). The model also expresses Maturana and Varela’s (1987) axiom that “all doing is knowing, and all knowing is doing” (p. 26) in that it represents mathematical understanding as the embodiment of all physical, mental, verbal, and written acts and locates Primitive Knowing as being the source from which all other mathematical knowings emerge (see Fig. 2).

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2 From “Growth in mathematical understanding: How can we characterize it and how can we represent it?,” by Pirie and Kieren (1994b). Copyright 1994 by Springer and Business Media and originally published as Fig. 1. Reprinted with permission of the authors and Springer and Business Media.
The model is not meant to be used as a tool with which one can categorize, level, or sequence forms of mathematical knowledge in the abstract but provides an alternative manner in which to conceptualize and describe the complexities inherent in mathematical understanding. Children cannot be said to be “at a level,” only “working at that level.” Outer layers in the model represent more generalized understanding of a topic (Pirie & Kieren, 1994b) and growth of mathematical understanding is not viewed as occurring through monodirectional outward pathways but ones that move both outwards and inwards. Depth and breadth of mathematical understanding then, increases as one moves outwards and inwards within the model; moving outwards opens possibilities for more generalized mathematical knowledge to emerge and movements inwards allow one to return to previous realms of knowing. Pirie and Kieren identify this process of return as essential to growth and term it “Folding Back.” Folding Back is defined, not as just the recollection of a mathematical experience or piece of information, but as providing a means by which a learner or group of learners can reconstruct, reintegrate, or re-evaluate known mathematics so that they may function in the outer layers with a “thicker” understanding (Kieren & Pirie, 1991; Martin, 1999; Towers, Martin, & Pirie, 2000).

3. The realms of Sam and Sammy’s understanding

We illustrate the understanding of Sammy and Sam, our two third-grade children, by walking the reader outwards through the Pirie–Kieren model of mathematical growth. It must be constantly borne in mind that we used the model to assess the children’s understanding of number and thus, are not suggesting to the reader that this is how Sam and Sammy’s understanding grew, but rather, how we organized the results of our analysis. We begin in the innermost layer of the model, “Primitive Knowing,” and move outwards to “Inventising,” offering what we can see of their current understanding from the data we gathered. During this discussion, we provide a brief description for each of the levels of knowing and excerpts from the videotaped session (total time: 23 min) or, the children’s written artifacts, to serve as examples that locate the different realms of knowing in which the children’s understanding existed. By doing so, we hope to illuminate the complexity

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3 From “Understanding for Teaching for understanding,” by Kieren (1990), Reprinted with permission of the author.
3.1. Primitive Knowing

The word “primitive” is used in its sense of “prime,” basic, prior, and not intended to convey any judgment as to the level of sophistication of mathematical or, indeed, any other knowledge the person possesses. It is everything a person brings “in his or her mind” to the current task except of course, understanding of the topic under scrutiny.

We mentioned earlier that Primitive Knowing is considered to be the source from which all mathematical understanding grows and because of this, not only does it comprise the informal and formal mathematics that one already embodies, but also the “real life” knowledge that one has. As observers, we can never know exactly what another person’s Primitive Knowing is. However, we can form interpretations based on the evidence that is made available to us through one’s physical, verbal, and written actions. In the following episode, we describe the primitive knowledge that we surmise Sammy and Sam bring to their task when they describe why 72 is an “even” and not an “odd” number:

Teacher: [reading what Sammy and Sam have written on their piece of paper as clue 4] It is an even number. Why does that make sense or, why does it describe your number? [72]
Sammy: Because if it [72] was a seventy-three, it would be odd because the seven and the three are odd... It’s [72] even because seventy-two can be divided by nine... it can be... divided into nine, into nine groups [points to “8 × 9 = 72” written on their piece of paper]
Teacher: Okay, so you’re saying, if seventy-two [is] even, it can be divided... [Sammy interjects]
Sammy: ...because eight times nine means eight groups of nine, equals seventy-two.
Teacher: Okay...
Sammy: It would be divided into equal groups.
Teacher: Now, you told me that if this [72] was seventy-three, not seventy-two...

Sam and Sammy: [interjecting in unison] An odd number!

Sammy: Because it can’t be divided by nine, it can’t be divided into nine groups.

Teacher: What if it [the number] was seventy-one or seventy-four? [no response] What makes an even number... even?

Sammy: You can divide it into equal groups.

Sam: You can count by twos. Two’s an even number [unfolds two of his fingers], four [unfolds two more fingers], six [lays his right hand over his left hand, we assume to ‘make’ more fingers], ...all the even numbers.

We see Sam as coming from an appropriate place of knowing when he explains that you can determine whether or not a number is even by counting by twos. That is, if you can “arrive” at the number starting at zero and skip-counting by twos, the number is considered to be “even.” However, it is unclear based on his verbal explanation and physical demonstration whether Sam’s understanding is that of a conceptual nature, a procedural one, or both.

Sammy, we believe, has folded back to his Primitive Knowing and related his mathematical knowledge to his existing understanding of ordinary language. He sees “even” as in “it all comes out even,” or “share it out evenly,” meaning fairly, exactly divisible, leaving no remainders, “divided into equal groups.” This is certainly also true of mathematically even numbers, but not the defining property. Furthermore, it is difficult to tell what Sammy means when he identifies 73 as being an “odd” because unfortunately, not only is 73 an odd and a prime number, its digits — 7 and 3 are also odd and prime. In hindsight, had he been asked to explain whether (and why) a number such as 49 was even or odd, this might have provided some insight into his understanding of odd numbers.

3.2. Image making

The first layer of coming to understand is when one performs actions — mental or physical — in order to create some “idea” of the new topic. Here, understanding grows from making distinctions through mathematical actions, on a base in Primitive Knowings. The intent of working at this level is to give rise to the creation of new mathematical “images,” which may exist in mental, verbal, written, or physical forms.

As mentioned earlier in the introduction of this paper, this activity was designed as an assessment task and what we were interested in was observing the images of number that the children already had. We can however, offer a possible explanation for how a particular image of Sammy’s, came to be.

Teacher: Did you use something other than symbols?

Sam: Yeah [points to clue 3 on their sheet that states, “It’s between 70 and 80”].

Sammy: Yeah. We used words.

Teacher: Can we move on to that [clue] and look at that?

Sammy: [reads clue 3] It is between seventy and eighty.

Sam: [reads clue 4] It is an even number.

Teacher: [interjects and covers up clue 4 so that they can only see clue 3] Okay. Why does this one make sense?

Sam: Because it is.

Sammy: Because it’s higher than seventy and lower than eighty.

Intrigued by Sammy’s response of 72 as being “higher” than 70 and “lower” than 80, we wondered if this was only a linguistic image or if he had actually made a visual image of “height.” In an interview following the conclusion of the session, the children’s teacher explained that the class would often play a game in which they would try to make the highest (or lowest) possible number by randomly drawing digits 0 through 9 out of an envelope and placing them as they drew them out in either the hundreds’, tens’, or ones’ column. After all of the digits had been drawn and a number was recorded, the teacher would ask the children to come up to the chalkboard and share their results. The rest of the class would then decide whether the number was “higher” or “lower” than the previous ones, and where it should be recorded in relation to the other numbers, creating a vertical number line. We see this as a teacher working to help children “make” a visual image and connect a linguistic one for the magnitude of a number in a way that a horizontal number line cannot.
3.3. **Image Having**

Image Having is a mathematical knowing that is already formed. To “have” a mathematical image means that understanding exists beyond the first “don’t need” boundary (see Fig. 1 for the bolded circles) in the Pirie–Kieren model. One knows something about the topic independent of the activities that led up to that knowing. The “Don’t’ Need” Boundaries are so called because one does not need to rely on the more specific, inner understandings that gave rise to the outer knowing. Understanding is said to be “connected,” however, if the knower can return to the inner layers of understanding when it proves necessary to change existing images that later show themselves to be inadequate or even incorrect. Put in simpler terms, to have a mathematical image is to “know” some piece of mathematics as a “matter of fact” (right or wrong!).

Sammy: Sam, let’s do it. Something about that secret number? What equals that number? Oh! [exclaims excitedly]
Sam: How ‘bout... divided! ... will make it equal...
Sammy: No, times! [i.e., multiplication] We know something that equals that number.
Sam: Yes [smiles and nods his head].
Sammy: What is it? [shakes his pencil at Sam and says...] Don’t say it, just let me think.

A few seconds later, Sammy records onto the paper, “$8 \times 9 = 72.$”

Sam: How do you know? [points to “$8 \times 9 = 72$” on the sheet of paper]
Sammy: [smiles] I know the times-table!

In this episode, Sam and Sammy search for an image that they already “have” in order to describe 72. Their conversation reveals that they do not intend to “create” or “make” an image for 72, but rather, they wish to Fold Back and collect an image that they already “have.” In this instance, the boys are not Folding Back to collect a specific mathematical image but simply one that will allow them to create the answer, 72. Sammy states the image he has of, “$8 \times 9 = 72,$” records it onto their sheet of paper, and exclaims that he just knows this to be “true” because of the “times-table!” He did not need to resort to the specific activities that gave rise to his understanding of multiplication as grouping.

3.4. **Property Noticing**

Once a person has made certain images, one is in a position to look at those images and make connections and distinctions among them. This is the layer of Property Noticing. It is a form of “standing back” and reflecting on one’s existing understanding in order to further that understanding.

After Sammy recorded the first clue (i.e., $8 \times 9 = 72$), he handed the paper to Sam, who then wrote “$144 \div 2 = 72.$” Later, their teacher asked Sam about this.

Teacher: How did you think of doing one hundred and forty-four divided by two equals seventy-two?
Sam: Because seventy-two plus seventy-two equals one hundred and forty-four. If we divided by two, we’d be cutting one hundred and forty-four in half.

We see Sam engaged in Property Noticing. His facility with the language he chooses to use reveals that he has formed connections among his images for the operations on these numbers. In this situation, repeated addition (possibly multiplication) is seen as the inverse of division, and halving as equivalent to dividing exactly by two. It is precisely Sam’s ability to make such varied distinctions and express the interrelationships amongst his images that illustrate qualities characteristic of Property Noticing. We do not, however, here have evidence that he has generalized his understanding, since he chooses to talk in terms of specific numbers, 144, 2, and 72.
3.5. Formalising

Mathematical Formalising occurs just outside of the second “don’t need” boundary in the Pirie–Kieren model; one does not need to use specific object-related images. It is in this realm of knowing that generalisations or common qualities about specific mathematical images are expressed as “for all” statements. If however in the previous episode, Sam had said something like “if you multiply and divide by the same number you get the number you started with,” we would have then been able to say, with evidence, that he was Formalising. The fact that he did not make a general statement does not, of course, mean that he had not Formalised, it simply means that he did not make this understanding available to us through some kind of observable demonstration at that moment.

We do, in fact, have evidence from an earlier part of the tape that Sammy had understanding that existed beyond the second “Don’t Need” Boundary. This is due to the serendipitous posing of the task by the teacher. Because the number, 72 was given to Sam and Sammy as a “secret number” and the clues that they produced were to be given to another pair of children to work on later, this had the effect of making the two boys whisper, which on one hand, made hearing the tape very difficult, but on the other hand, led to Sammy pointing with his finger at the 72 and saying to Sam, before calculating and Sam writing “144,”

Sammy: That number plus that number divided by two equals that number.

Even though he is not using formal algebraic language such as \((n+n)/2=n\) we do however, see Sammy as working within the level of Formalising because he is thinking in terms of “any” number and does not need to specifically calculate \(72+72=144\), then \(144/2=72\) in order to make his statement. With hindsight, based on the responses to the teacher at the end of the session, we can also interpret the next part of the whispered conversation we heard at this earlier point.

Sam: [Obviously to preserve secrecy, puts his fingers to his lips] sssh... sssh [and he begins counting from 70 and unfolds his pinkie finger to monitor his adding of groups of ten.]

Sammy: Oh, I know [whispers in Sam’s ear].

Sam: Let me count. Seventy...

Sammy: [whispers something]... multiplied by two...

Sam Folds Back from their Formalized statement to work at the Image Having level with the specific strategy of skip counting by tens for 72 + 72. From his actions and Sammy’s comment, it is evident that they do, in fact, also relate repeated addition and multiplication as we surmised above. This incident clearly demonstrates the power of working constantly with the original data as it was following the talk with the teacher that we were able to search back in order to make sense of the boys’ whisperings and verify their understanding.

3.6. Observing

Observing entails reflecting on and coordinating one’s formal mathematical activities. In other words, Observing is to Formalising as Property Noticing is to Image Having. We have seen that Sammy and Sam have confident formalised understandings for the operations of addition, multiplication, and division and so we looked at the data to see whether their understanding had perhaps gone beyond this to a situation where they believed in some “theorems” about the operations. Clearly one would not expect children of this age to be using theorems or proofs, but the Pirie–Kieren theory posits that even young children can reach sophisticated levels of mathematical thinking in topics that they understand. We were alerted by an interesting comment from Sammy, following the writing of their first two clues.

Sammy: Oh, let’s say something about it [72]. Let’s not do any of the sums.

We conjectured that Sammy was intending that they offer their classmates some of the “properties” of their secret number rather than just give them calculations to do; he saw this as more interesting or harder than simple arithmetic since they wanted to challenge the other children and not make it too easy to know what their number was. Their next three clues were written as:
“3. It’s between 70 and 80.

4. It is a (sic) even number.

5. It is a two digit number.”

At this point, they decided on yet another form to express what they knew about 72. They created a diagram of seven circles containing ten crosses and one circle containing two crosses for the sixth clue (see Fig. 3). Their seventh clue was a verbal version of the diagram:

“7. 7 groups of 10 and 1 group of 2 makes (sic) this number.”

Sammy then wrote:

“8. 10 + 10 + 10 + 10 + 10 + 10 + 10 + 2 = 72” and later after some discussion as to what to do for another clue, Sam wrote:

“9. 144 − 10 − 10 − 10 − 10 − 10 − 10 − 2 = 72.”

An episode of Observing was revealed when the teacher asked Sam about clue 9. Again, the video data was essential for the revelation of this sophistication of understanding, since much of the evidence is drawn from Sammy and Sam’s hand movements. Children do not need to be linguistically fluent to think in deep mathematical ways.

Teacher: How’d you come up with this? [pointing to clue 9.]

Sam: I can read the plus [points to the clue 8, which reads 10 + 10 + 10 + 10 + 10 + 10 + 10 + 2 = 72]...

Sammy: 42 (sic) [presumably meaning 72] And if you minus one of the, take away one of the seventy-twos, but actually don’t say something [such as “72”], do it another way, like this one [sweeps his finger over “10 + 10 + 10 + 10 + 10 + 10 + 10 + 2 = 72”] you get 72. Equals another 72 [points to “= 72” in “144 − 10 − 10 − 10 − 10 − 10 − 10 − 2 = 72”].

We suspected that Sammy was trying to explain that rather than write 144 − 72 = 72, he had written 144 − (10 + 10 + 10 + 10 + 10 + 10 + 10 + 2) = 72. He did not write it in this fashion — probably because he has not learned about the symbolism for brackets — but it is clear that neither is he talking about subtracting 10, then subtracting 10, then subtracting 10, etc. From our perspective, we see Sammy using a specific case of the distributive law over addition:

−(10 + 10 + 10 + 10 + 10 + 10 + 10 + 2) = −10 − 10 − 10 − 10 − 10 − 10 − 10 − 2. In this way, we see him Observing a feature of his formal understanding of addition even though he has not stated it as a “theorem.” The teacher does not react to this explanation and the boys continue to explain what they did:

Sam: Seventy-two plus seventy-two equals one hundred and forty-four. [stretches his thumb and fingers around either end of the “−10 − 10 − 10 − 10 − 10 − 10 − 2” and says] Seventy-two.

Sammy: [interjects] Sam just took another seventy-two by not saying it.

Teacher: [repeating what Sammy had said but in a louder voice so that Sam could hear]. Sam just took another seventy-two by not saying it?

Sammy: By not... writing it [72] down [sweeping his fingers in a circle over top of “144 − 10 − 10 − 10 − 10 − 10 − 2 = 72”].

Sam: By not... like... one hundred and forty-four minus... you minus one hundred and forty-four by seventy-two [sweeps finger across “144 − 10 − 10 − 10 − 10 − 10 − 10 − 2 = 72.”] Because... because... [points to 144 ÷ 2 = 72].

Sammy: [interjects] But you also minused... Sam... Sam also minused seventy-two but he also... one hundred forty-four minus seventy-two equals seventy-two. He also minused the seventy two.

They continue patiently to go over this idea of “minusing 72,” and the 72 coming from clue 8, for several more minutes, but at no point do they say “144 minus 10, minus 10, etc.” To us it is clear that they think of subtracting 72 in the form 10 + 10 + 10 + 10 + 10 + 10 + 10 + 2 as they have written it in statement 8, but they are still talking in terms of a specific example.

A while later, the teacher interviewed the boys about this work. Sam and Sammy returned to clue 8 but this time they worked with other numbers (i.e., 85, 37, 1010, and 9). They expressed each number as they had done previously “by not saying it” and doing “it another way” by recording the number as an addition equation (see Fig. 4).
The teacher then asked Sam and Sammy whether they thought what they had done for clue 9 could be done for any number.

Sammy: I’d say yes, most of them...
Sam: [interjects] Most probably. [begins to speak to Sammy] Pick a number. Pick a number and I’ll try.
Sammy: Okay, try the first number we did [referring to what they had written about ‘85’ in 11 + 11 + 11 + 11 + 11 + 11 + 11 + 1 = 85 earlier in the interview], eighty-five and...
Sam: [interjects] Just pick any number. Just pick a number.

The boys then spend some time Folding Back and working with specific numbers, talking to each other as they work. Trying to provoke them to work at an outer layer, the teacher intervenes:

Teacher: Sammy, [when you were talking to Sam just now] you said that you thought it would work for any number because it was almost opposite. What did you mean by that?
Sammy: It’s subtracting instead of adding [referring to what he had recorded as: 11 + 11 + 11 + 11 + 11 + 7 + 1 = 85].
Sam: ... to do this... maybe the number you pick, you add it, you add it with the same number that you... picked... you get the big number... .

Sam then records onto a piece of a paper (see Fig. 5):

Sammy: [interjects] you times it by two, minus it [the number].
Sam: Like eighty-five plus eighty-five equals one hundred and seventy, and you minus it [sweeps his finger across “−11 − 11 − 11 − 11 − 11 − 11 − 7 − 1”].
Sammy: [sweeps his finger across “11 + 11 + 11 + 11 + 11 + 11 + 11 + 7 + 1”].

To get rid of the extra half.
This definitely confirms that Sam and Sammy have formalised their understanding and gives even stronger evidence that they are observing their formalisations of working with natural numbers.

Fig. 4. Sam and Sammy’s written work related to clue 8 using the values of 85, 37, 1010, and 9.

Fig. 5. Sam’s written work that demonstrates their ‘add the number by itself/multiply the number by two and then minus it by “not saying it”’ “rule” related to clue 9 and how it applies to other values, such as in this case, 85.
3.7. Structuring

Structuring involves being able to explain or theorise one’s formal observations in terms of a logical structure. It is a layer at which general observations on formalisations of the topic under consideration are subsumed into a mathematical structure and no longer need to be treated as specific cases of that structure. For these children, it would possibly be seeing natural numbers as part of the set of whole numbers and knowing that laws such as the distributive law hold good over the entire set. From the videotape we could not determine whether the children were able to move their understanding over the third “Don’t Need” Boundary and out to the level of Structuring. We had no evidence of their understanding, if any, of zero or further still, negative numbers. Since research has found little evidence of children in school being encouraged to observe and structure their formal understandings, this layer is not yet well understood and is being explored by the theory authors and others.

3.8. Inventising

Inventising requires one to break away from the preconceptions that brought forth earlier understanding in order to pose new questions that may give rise to the growth of a totally different concept. Given a structured understanding of integers, a child might wonder whether there were “numbers” between the integers that obeyed the same laws. Given current curriculum frameworks and their associated topic-teaching hierarchies, it is unlikely that children are given the opportunity to inventise in the classroom. New topics are generally introduced as such, and in this case can lead to many children not understanding that fractions and decimals are indeed “numbers” too. For many children, even in early secondary school “number” means only “whole number.” Fractions are fractions — something else!

3.9. Folding Back and Connected Understanding

We have, in passing, alluded to examples of these features, but since this activity was an assessment task, we did not expect to necessarily see much evidence of Folding Back. It is a vital part of coming to understand at any level, but these children were not expecting to learn anything new from the activity. Had the teacher perhaps challenged the boys on their understanding of the distributive law, inviting them to prove in some way what they were using, then we might well have seen them Folding (right) Back to Image Having to work on the idea, or we may have seen them provoked by the challenge and moving out to offer a form of proof.

4. Conclusion

By using the Pirie–Kieren dynamical theory and locating Sam and Sammy’s forms of knowing within it, we have attempted to show the reader a small part of the richness that is embedded in these two children’s understanding of natural numbers. We conceive that “good” mathematical understanding entails the integration of informal and formal mathematical knowledge; knowledge that is flexible and fluid. The assessment task, “What do you know about 72?” and the analysis that followed, revealed that these children indeed, have such an understanding and are probably ready to move on to work with notions of integers. As our research continues and we follow the work of these two boys, we will explore their individual and collective mathematical growth related to the understanding of number. In fact, data we have subsequently collected has caused us great excitement! But that is another story. For now, we feel we have a good understanding for the ground in which their growth of mathematical understanding is being rooted.

References