Imagine you work for a major petroleum company as a technical support representative for all of the gas stations in a certain region. Gasoline is stored in underground cylindrical tanks just below the surface of the station. The tanks are positioned with their circular bases perpendicular to the surface. An opening on the top center of the tank is used to fill and measure the gasoline in the tank. A pipe at the bottom of the tank transfers gasoline to the pumps.

A dipstick measures the amount of gasoline left in the storage tank. It is lowered through the opening in the storage tank until it touches the bottom of the tank. Then it is removed. The wet part of the stick indicates how many gallons of gasoline are left in the storage tank according to the markings on the stick. It is similar to the oil dipstick used in a car.

The owner of one of the gas stations that you service has misplaced his dipstick for one of his tanks. He could order a replacement stick but since the tank is an odd size, the dipstick is very expensive. He asks you to design a stick with the appropriate scale marked on it.

In this Investigation, you will begin to research the problem by experimenting with coffee cans and wooden chopsticks. The coffee cans will represent the gas tanks, and the chopsticks will represent dipsticks.

**Materials Needed**
- 2 sizes of coffee cans
- Wooden chopsticks
- Duct tape
- Large nail
- Hammer
- Metal snips
- Metric ruler
- Funnel
- Metric measuring cup

**SET UP:**

1. Make a model of a gas tank by using a coffee can with the plastic lid taped securely onto the can with duct tape.
2. Punch a hole in the side of the coffee can using a nail and hammer. Using a pair of metal snips, make the hole big enough to insert a funnel.
**Coffee Can Experiment**

<table>
<thead>
<tr>
<th>Radius of can:</th>
<th>Length of can:</th>
<th>Capacity of can:</th>
</tr>
</thead>
<tbody>
<tr>
<td>water in can (mL)</td>
<td>measure on dipstick (cm)</td>
<td>% of can filled</td>
</tr>
<tr>
<td>50 mL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 mL</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Copy the chart above onto a sheet of paper.

4. Collect Data:

4. Measure the radius of the lid and the length of the coffee can. Record these data in your chart.

5. Use a funnel and a metric measuring cup to pour 50 mL of water into the coffee can through the hole on its side. Lower the chopstick into the hole until it touches the bottom and pull it out. With a metric ruler, measure the distance on the chopstick that is wet and record the measurement in your chart.

6. Repeat this process, adding 50 mL at a time, until the coffee can is full. Did you use all of the last 50 mL? What is the capacity of the coffee can in mL? Record this value in your chart.

7. Estimate how much water is in the can when it is one-half full, one-fourth full, and three-fourths full. Make a note of these numbers.

8. For each 50-mL increment, calculate the ratio of the amount of water in the coffee can to the capacity of the coffee can. Convert these ratios to percents of a full coffee can. Record these data in your chart. Were your estimates correct for the amount of water when the can is one-half full, one-fourth full, and three-fourths full?

9. For each increment, calculate the ratio of the measure on the dipstick to the total length of the dipstick. Convert these ratios to percents of a fully-covered dipstick. Record these data in your chart.

10. Make a drawing to illustrate the measurement scale for a dipstick for this coffee can. Include markings for one-fourth, one-half, and three-fourths full.

11. How does the height of the water change as the amount of water in the can changes? Determine the change in the percent of water in the tank at each 50-mL increment. Record these data in your chart.

Make an Investigation Folder in which you can store all of your work on this investigation for future use. Be sure to keep your chart and materials in your Investigation Folder.

You will continue working on this Investigation throughout Chapters 8 and 9.

*Investigation: Fill It Up!*
What You’ll Learn

- To evaluate polynomial functions, and
- To identify general shapes of the graphs of polynomial functions.

Why It’s Important

You can use polynomial functions to solve problems involving biology and energy.

**Calvin and Hobbes**

Calvin has to give up his ambition of migrating with the wildebeests when he learns that they live on another continent. The Serengeti Plain is a wildlife reserve in Africa where herds of wildebeests, or gnus, roam. The population of wildebeests on the Serengeti Plain can be described by the function $f(x) = -0.125x^5 + 3.125x^4 + 58,000$, where $x$ represents the number of years since 1990. The graph at the right shows this function. The expression $-0.125x^5 + 3.125x^4 + 58,000$ is a **polynomial in one variable**. You will find the population of wildebeests in Example 2.

**Definition of a Polynomial in One Variable**

A polynomial of degree $n$ in one variable $x$ is an expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$ represent real numbers, $a_0$ is not zero, and $n$ represents a nonnegative integer.

In Chapter 2, linear functions, which are degree 1, were identified and graphed. In Chapter 6, quadratic functions, which are degree 2, were identified and graphed. In general, the degree of a polynomial in one variable is determined by the greatest exponent of its variable.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Expression</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$a_0$</td>
<td>0</td>
</tr>
<tr>
<td>Linear</td>
<td>$ax + b$</td>
<td>1</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$ax^2 + bx + c$</td>
<td>2</td>
</tr>
<tr>
<td>Cubic</td>
<td>$ax^3 + bx^2 + cx + d$</td>
<td>3</td>
</tr>
<tr>
<td>General</td>
<td>$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Remember that $4 = 4x^0$ and $x + 8 = x^1 + 8x^0$. 

478 Chapter 8 Exploring Polynomial Functions
Example 1

Determine if each expression is a polynomial in one variable. If so, determine its degree.

a. \(6x^4 + 3x^2 + 4x - 8\)

This is a polynomial in one variable, \(x\). The degree is 4.

b. \(9x^3y^5 + 2x^2y^6 - 4\)

This is not a polynomial in one variable. It contains two variables, \(x\) and \(y\).

c. \(t^{-3} + 4t^2 - 1\)

This is not a polynomial, because the variable has a negative exponent.

d. \(5x^7 + 3x^2 + \frac{2}{x}\)

This is not a polynomial, because the term \(\frac{2}{x}\) cannot be written in the form \(x^n\), where \(n\) is a nonnegative integer.

When a polynomial equation is used to represent a function, the function is a **polynomial function**. For example, the equation \(f(x) = 4x^2 - 5x + 2\) describes a quadratic polynomial function, and the equation \(p(x) = 2x^3 + 4x^2 - 5x + 7\) describes a cubic polynomial function. These and other polynomial functions can be defined by the following general rule.

### Definition of a Polynomial Function

A polynomial function of degree \(n\) can be described by an equation of the form \(P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n\) where the coefficients \(a_0, a_1, a_2, \ldots, a_{n-1}, \text{ and } a_n\) represent real numbers, \(a_0\) is not zero, and \(n\) represents a nonnegative integer.

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Remember that if \(f(x)\) is the function and 4 is an element in the domain, the corresponding element in the range is \(f(4)\). To find \(f(4)\), evaluate the function for \(x = 4\).

Example 2

Refer to the application at the beginning of the lesson. Use the polynomial function to estimate the population of wildebeests in 1995.

\(x\) represents the number of years since 1990, 1995–1990 or 5.

\[f(x) = -0.125x^5 + 3.125x^4 + 58,000\]

\[f(5) = -0.125(5)^5 + 3.125(5)^4 + 58,000\] Replace \(x\) with 5.

\[= -390.625 + 1953.125 + 58,000\text{ or }59,562.5\ Evaluate.\]

Therefore, in 1995, there were approximately 59,562 wildebeests.

Example 3

a. Find \(p(a + 2)\) if \(p(x) = x^3 - 2x + 1\).

\[
p(a + 2) = (a + 2)^3 - 2(a + 2) + 1 \quad \text{Substitute } a + 2 \text{ for } x.
\]

\[
= a^3 + 6a^2 + 12a + 8 - 2a - 4 + 1
\]

\[
= a^3 + 6a^2 + 10a + 5 \quad \text{(continued on the next page)}
\]
Remember that the x-coordinate of a point at which the graph crosses the x-axis is called a zero of the function. On the coordinate plane, these zeros are real numbers.

b. Find \(-2p(\alpha) + p(\alpha + 1)\) if \(p(x) = x^3 + 3x^2 - 5\).

\[
-2p(\alpha) + p(\alpha + 1) = [-2(\alpha^3 + 3\alpha^2 - 5)] + [(\alpha + 1)^3 + 3(\alpha + 1)^2 - 5]
\]
\[
= -2\alpha^3 - 6\alpha^2 + 10 + \alpha^3 + 3\alpha^2 + 3\alpha + 1 + 3(\alpha^2 + 2\alpha + 1) - 5
\]
\[
= -2\alpha^3 - 6\alpha^2 + 10 + \alpha^3 + 3\alpha^2 + 3\alpha + 1 + 3\alpha^2 + 6\alpha + 3 - 5
\]
\[
= -\alpha^3 + 9\alpha + 9
\]

The graphs of several polynomial functions are shown below. Notice how many times the graph of each function intersects the x-axis. In each case, this is the maximum number of real zeros the function may have. How does the degree compare to the maximum number of real zeros?

**Constant function**

\[
f(x) = 2
\]
Degree 0

**Linear function**

\[
f(x) = \frac{3}{2}x - 3
\]
Degree 1

**Quadratic function**

\[
f(x) = x^2 + 2x - 3
\]
Degree 2

**Cubic function**

\[
f(x) = x^3 - 5x + 2
\]
Degree 3

**Quartic function**

\[
f(x) = x^4 - 3x^3 - 2x^2 + 7x + 1
\]
Degree 4

**Quintic function**

\[
f(x) = x^5 - 5x^3 + 4x
\]
Degree 5

These are the general shapes of the graphs for polynomial functions with degree greater than 0 and positive leading coefficients. The leading coefficient is the coefficient of the term with the highest degree. So, for the function \(f(x) = 5x^3 + 2x^2 + 8x - 1\), the leading coefficient is 5.

Notice that the simplest polynomial graphs have equations in the form \(f(x) = x^n\), where \(n\) is a positive number. Note the general shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions.

- **Even-degree polynomial functions**
  - \(f(x) = x^4\)
  - \(f(x) = x^2\)
  - \(f(x) = x^0\)

- **Odd-degree polynomial functions**
  - \(f(x) = x^3\)
  - \(f(x) = x\)
  - \(f(x) = x^5\)
Note that the even-degree functions are tangent to the x-axis at the origin. When this happens, the function has two zeros or roots that are the same number. For example, \(x^2 - 6x + 9 = 0\) can be factored as \((x - 3)(x - 3) = 0\). So 3 is the root of the equation.

An even-degree function may or may not intersect the x-axis, depending on its location in the coordinate plane. If it does not intersect the x-axis, its roots are all imaginary. An odd-degree function always crosses the x-axis at least once. Why?

**Example 4**

Determine if each graph represents an odd-degree function or an even-degree function. Then state how many real zeros each function has.

(a) ![Graph of a function](image)

(b) ![Graph of a function](image)

(c) ![Graph of a function](image)

<table>
<thead>
<tr>
<th>Graph</th>
<th>left-most y values</th>
<th>right-most y values</th>
<th>degree of function</th>
<th>times graph crosses x-axis</th>
<th>number of real zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>positive</td>
<td>positive</td>
<td>even</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>b.</td>
<td>negative</td>
<td>positive</td>
<td>odd</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>c.</td>
<td>positive</td>
<td>positive</td>
<td>even</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

In Chapter 2, you studied families of graphs of linear equations. In Chapter 6, you studied families of parabolas. Some families of graphs of polynomial equations are shown below. The equation below each graph is the equation of the parent graph for that family.

- \(y = 4x^2\)
- \(y = x^2\)
- \(y = -x^2\)
- \(y = -4x^2\)
- \(y = x^4\)
- \(y = x^4 + 2\)
- \(y = (x+2)^3\)
- \(y = x^6\)
- \(y = (x-4)^5\)
- \(y = x^5\)

**CHECK FOR UNDERSTANDING**

**Communicating Mathematics**

Study the lesson. Then complete the following.

1. Refer to Example 4. What appears to be the degree of each function? 4, 3, 2
2. Describe the characteristics of the graphs of odd-degree and even-degree polynomial functions whose leading coefficients are positive. See margin.
3. State how many real zeros are possible for each polynomial function.

   a. quartic 4  
   b. linear 1  
   c. quadratic 2  
   d. quintic 5  
   e. cubic 3
4. **Sketch** the graph of an odd-degree function with a positive leading coefficient and three real roots. See margin for sample graph.

5. **You Decide** Carlos explains to his friend Zach, “The graphs of odd-degree polynomial functions always intersect the x-axis an odd number of times, and the graphs of even-degree functions always intersect the x-axis an even number of times.” Zach doesn’t believe this is always the case. Who is correct? Give an example to support your answer. See margin.

6. Look at the family of graphs at the right for the function \( f(x) = x^3 \). Investigate the relationship between the similar functions and their graphs. See margin.

**Guided Practice**

8. No, the polynomial contains two variables, \( a \) and \( b \).

11. quintic, 5, 5

13. quartic, 4, 4

**Find the degree of each polynomial in one variable. If it is not a polynomial in one variable, explain why.**

7. \( 4t^3 + 8t^2 + 2t - 1 \)  3  8. \( 9ab^2 + 4ab + 3 \)  9. \( 7x^3 - 8x^5 + 8x - 7 \)

**Identify each polynomial function as linear, quadratic, cubic, quartic, or quintic. State the degree and how many real zeros are possible.**

10. \( f(x) = 6x^3 + 8x + 7 \)  cubic, 3, 3

11. \( f(x) = 3x^5 + 7x^4 - 5x^3 + 6x + 9 \)

12. \( f(x) = 2x + 5 \)  linear, 1, 1

13. \( f(x) = 5x^3 + 6x^2 - 8x^4 - 10x + 1 \)

**Match the polynomial and its functional value.**

14. \( p(x) = 3x^2 + 4x + 5 \)  b

15. \( p(x) = x^4 - 7x^3 + 8x - 6 \)  a

16. \( p(x) = 7x^2 - 9x + 10 \)  \( \text{d} \)

17. \( p(x) = 4x^3 - 2x^2 - 6x + 5 \)  c

18. Refer to the graph at the right.

a. Determine whether the degree of the function is even or odd. **even**

b. How many real zeros does the polynomial function have? 3

**Find \( p(2) \) and \( p(-1) \) for each function.**

19. \( p(x) = 2x^2 + 6x - 8 \)  12, -12

20. \( p(x) = -3x^4 + 1 \)  -47, -2

**Find \( f(x + h) \) for each function.**

21. \( f(x) = 2x - 3 \)  \( 2x + 2h - 3 \)

22. \( f(x) = 4x^2 \)  \( 4x^2 + 8xh + 4h^2 \)

23. **Energy** The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function \( P(s) = \frac{s^3}{1000} \) where \( s \) represents the speed of the wind in kilometers per hour. Find the units of power generated by a windmill when the wind speed is 25 kilometers per hour. 15,625 units

482 Chapter 8 Exploring Polynomial Functions
EXERCISES

Practice

Determine whether the degree of the function represented by each graph is even or odd. How many real zeros does each polynomial function have?

24. \[ f(x) \]
25. \[ f(x) \]
26. \[ f(x) \]

Find \( p(3) \) and \( p(-2) \) for each function.

27. \( p(x) = 5x + 6 \quad 21, -4 \)
28. \( p(x) = x^2 - 2x + 1 \quad 4, 9 \)
29. \( p(x) = 2x^3 - x^2 - 3x + 1 \quad 37, -13 \)
30. \( p(x) = x^5 - x^2 \quad 234, -36 \)
31. \( p(x) = -x^4 + 53 \quad -28, 37 \)
32. \( p(x) = x^5 + 5x^4 - 15x^2 - 8 \quad 505, -20 \)

Find \( f(x + h) \) for each function.

33. \( f(x) = x + 2 \quad x + h + 2 \)
34. \( f(x) = x^2 - 2x + h - 4 \)
35. \( f(x) = 5x^2 \quad 5x^2 + 10xh + 5h^2 \)
36. \( f(x) = x^2 - 2x + 5 \)
37. \( f(x) = 3x^2 + 7 \quad 3x^3 + x \)

Find \( 4[p(x)] \) for each function.

38. \( p(x) = x^2 + 5 \quad 40. \quad 24x^3 - 16x^2 + 8 \)
39. \( p(x) = 6x^3 - 4x^2 + 2 \quad 41. \quad \frac{x^3}{4} + \frac{x^2}{16} - 2 \)

Find an equation for each graph.

42. \[ f(x) \]
43. \[ f(x) \]
44. \[ f(x) \]

45. Sketch a graph of a polynomial function \( f(x) \) that has the indicated number and type of zeros. See Solutions Manual for sample graphs.
   a. 5 real 
   b. 3 real, 2 imaginary 
   c. 4 imaginary

Find \( 2p(a) + p(a - 1) \) for each function.

46. \( p(x) = 4x + 1 \quad 47. \quad p(x) = x^2 + 3 \quad 48. \quad p(x) = x^2 - 5x + 8 \)
46. \( 12a - 1 \quad 3a^2 - 2a + 10 \quad 3a^2 - 17a + 30 \)

Find \( 2[f(x + 3)] \) for each function.

49. \( f(x) = 2x + 9 \quad 50. \quad f(x) = x^2 - 6 \quad 51. \quad f(x) = x^2 + 3x + 12 \)
49. \( 4x + 30 \quad 2x^2 + 12x + 6 \quad 2x^2 + 18x + 60 \)

52. Sketch a graph that matches each description. Write an equation for the graph of each function. See Solutions Manual for sample graphs.
   a. a quadratic function with zeros at -1 and 2. \( f(x) = x^2 - x - 2 \)
   b. a cubic function with zeros at -2, -1, and 2. \( f(x) = x^3 + x^2 - 4x - 4 \)
   c. a quartic function with zeros at -2, -1, 1, and 2. \( f(x) = x^4 - 5x^2 + 4 \)
   d. a quintic function with zeros at -2, -1, 0, 1, and 2. \( f(x) = x^5 - 5x^3 + 4x \)
53. There is no real number \( x \) that can make the equation \( G = x^4 + x^2 + 1 \) true.

54. The graph of the polynomial function \( f(x) = ax(x - 4)(x + 1) \) goes through the point at \((5, 15)\).
   a. Find the value of \( a \). \( \frac{1}{2} \)
   b. Sketch the graph of the function. See margin.

55. Biology The intensity of light emitted by a firefly can be determined by the polynomial function \( L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3 \), where \( t \) is the temperature in Celsius and \( L(t) \) is the light intensity in lumens. If the temperature is 30°C, find the light intensity. 109 lumens

56. Patterns If you look at a cross section of a honeycomb, you see a pattern of hexagons. This pattern has one hexagon surrounded by six more hexagons. Surrounding these is a third “ring” of 12 hexagons, and so on. Assume that the pattern continues.
   a. Find the number of hexagons in the 4th ring. 18
   b. Make a table that shows the total number of hexagons in the first ring, the first two rings, the first three rings, and the first four rings.
   c. Show that the polynomial function \( h = 3r^2 - 3r + 1 \) gives the total number of hexagons when \( r = 1, 2, 3 \).
   Identify the domain and range of this function. See margin.
   d. Use the equation to find the total number of hexagons in a honeycomb with 12 rings. 397

57. Seismology Two tracking stations have detected an earthquake. The first station determined that the epicenter was 25 miles away. The second station determined that the epicenter was 42 miles away. If the first station is located at the origin and the second station is 50 miles due east of the first station, where could the epicenter have been? (Lesson 7-7)

58. Write the standard form of the equation \( x^2 + 4y^2 = 4 \). Graph the equation and state whether the graph is a parabola, a circle, an ellipse, or a hyperbola. (Lesson 7-6)

59. Find two numbers whose difference is -40 and whose product is a minimum. (Lesson 6-4) -20, 20

60. Solve \( \sqrt{n} + 12 - \sqrt{n} = 2 \). (Lesson 5-8) 4

61. Biology The average human has between 1,600,000 and 1,700,000 sweat glands, mostly located in the palms of the hand and soles of the feet. Express both of these figures in scientific notation. (Lesson 5-1)

62. Determine the dimensions of the product \( A_{3 \times 2} \cdot B_{2 \times 3} \). (Lesson 4-3) \( 4 \times 3 \)

63. Find \( \begin{bmatrix} -9 & 6 \\ 5 & 19 \end{bmatrix} - \begin{bmatrix} -3 & 18 \\ -4 & 12 \end{bmatrix} \). (Lesson 4-2) \( \begin{bmatrix} -6 & -12 \\ 9 & 7 \end{bmatrix} \)

64. In which octant does the point at \((7, -2, 9)\) lie? (Lesson 3-7) 2

65. Solve the system of equations by using either the substitution or elimination method. (Lesson 3-2) \((4, -28)\)
   \[ \begin{align*}
   2x - y &= 36 \\
   3x - \frac{1}{2}y &= 26
   \end{align*} \]

66. Evaluate \( 2 \left| -3x \right| - 9 \) if \( x = 5 \). (Lesson 1-5) 21

484 Chapter 8 Exploring Polynomial Functions
The Remainder and Factor Theorems

**APPLICATION**

**Sports**

A javelin is usually thrown from about shoulder height, around 5 feet off the ground. It is not unusual for the javelin to start off with an upward velocity of about 74 feet per second. Based on this information, one can determine that the height of a javelin $t$ seconds after it is thrown can be described by the function $h(t) = -16t^2 + 74t + 5$, if the effect of air resistance is ignored. The 16 in the function is associated with the strength of the Earth's gravity; if the javelin were thrown on another planet, there would be a different number in the equation.

The graph of $h(t)$ is shown at the right. Notice that when $t = 0$, the value of $h(t)$ is 5. Suppose we find the height of the javelin after 3 seconds.

$h(t) = -16t^2 + 74t + 5$

$h(3) = -16(3)^2 + 74(3) + 5$

$= -144 + 222 + 5$

$= 83$

After 3 seconds, the height of the javelin is 83 feet.

Divide the polynomial in the function by $t - 3$, and compare the remainder to $h(3)$.

**Method 1: Long Division**

\[
\begin{array}{c|cccc}
\multicolumn{1}{r}{t - 3} & -16t^2 & + 74t & + 5 \\
\cline{2-5}
& -16t^2 & + 48t & \\
\multicolumn{1}{r}{+ 26t} & + 5 \\
\multicolumn{1}{r}{- 26t} & - 78 \\
\multicolumn{1}{r}{83} &
\end{array}
\]

**Method 2: Synthetic Division**

\[
\begin{array}{c|cccc}
3 & -16 & 74 & 5 \\
\cline{2-5}
\multicolumn{1}{r}{-48} & 78 \\
\multicolumn{1}{r}{-16} & .26 & 83 \\
\end{array}
\]

Notice that the value of $h(3)$ is the same as the remainder when the polynomial is divided by $t - 3$. This illustrates the **remainder theorem**.

### The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$, and

\[
dividend = quotient \cdot divisor + remainder
\]

where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$. 