Roots and Zeros

What You'll Learn

• To find the number and type of zeros of a polynomial function.

Why It's Important

You can qualify the number of roots for polynomials that model situations in marketing and physiology.

Application

Marketing

Mrs. Botti's French class is selling long-stemmed roses to raise money for their trip to France. The students are making boxes in which they will deliver the flowers. They want the boxes to have square ends (width and height the same), but the length should be 12 inches longer than the width so the very long flowers will fit. They want the volume of each box to be 256 cubic inches, so that each box can hold enough moistened packing material to keep the roses fresh. Find the dimensions for a box that satisfy all these requirements.

We can find the dimensions of the box by writing a polynomial equation. Then we can use the factor theorem and synthetic substitution. First define each dimension of the box in terms of the width \( w \).

\[ w = \text{width} = \text{height} \quad w + 12 = \text{length} \]

Then let \( V(w) \) be the function defining the volume. \( w \)

\[ V(w) = w \cdot w \cdot (w + 12) \quad \text{volume} = \ell \times w \times h \]

\[ 256 = w^3 + 12w^2 \]

Substitute 256 for \( V(w) \) and multiply. Subtract 256 from each side.

\[ 0 = w^3 + 12w^2 - 256 \]

Study the chart below. We will use a shortened form of synthetic substitution for several values of \( w \) to search for the solutions to \( 0 = w^3 + 12w^2 - 256 \). The values for \( w \) are in the first column of the chart. Besides each value is the last line of the synthetic substitution. Recall that the first three numbers are the coefficients of the depressed polynomial. The last number in each row is the remainder.

<table>
<thead>
<tr>
<th>( w )</th>
<th>1</th>
<th>12</th>
<th>0</th>
<th>-256</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>13</td>
<td>-243</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>14</td>
<td>-200</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>15</td>
<td>-121</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>16</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>17</td>
<td>169</td>
<td></td>
</tr>
</tbody>
</table>

A remainder of 0 occurs when \( w = 4 \). This means that \( w - 4 \) is a factor of the polynomial. The depressed polynomial is \( w^2 + 16w + 64 \).

The polynomial \( w^3 + 12w^2 - 256 \) can be factored as \((w - 4)(w^2 + 16w + 64)\). The trinomial \( w^2 + 16w + 64 \) can be further factored as \((w + 8)(w + 8)\) or \((w + 8)^2\). Thus, the solutions of the equation \( 0 = (w - 4)(w + 8)^2 \) are \( w = 4 \) and \( w = -8 \). Since negative widths are not possible when designing a box, the width and height of the box should be 4 inches, and the length should be 12 or 16 inches. Do these dimensions produce the correct volume?

In Chapter 6, you learned that a zero of a function \( f(x) \) is any value \( c \) such that \( f(c) = 0 \). This zero is also a root, or solution, of the equation formed when \( f(x) = 0 \). When the function is graphed, the real zeros of the function will be the \( x \)-intercepts of the graph.
In the equation \( w^3 + 12w^2 - 256 = 0 \), the roots are 4 and \(-8\). The graph of \( V(w) = w^3 + 12w^2 - 256 \) touches or crosses the horizontal axis at those two points.

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero have at least one root in the set of complex numbers. This is the **fundamental theorem of algebra**.

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

The following corollary of the fundamental theorem of algebra is an even more powerful tool for problem solving.

A polynomial equation of the form \( P(x) = 0 \) of degree \( n \) with complex coefficients has exactly \( n \) roots in the set of complex numbers.

Notice that \( w^3 + 12w^2 - 256 = 0 \) appears to have only two roots, even though it is a third-degree equation. However, remember that the factored version of the equation was \((w - 4)(w + 8)^2 = 0\). The fact that \( w + 8 \) appears twice among the factors of \( w^3 + 12w^2 - 256 \) means \(-8\) is a "double root." Since it is understood that \(-8\) is counted twice among the roots of the equation, we know that \( w^3 + 12w^2 - 256 = 0 \) really has three roots: \(-8, -8, \) and \(4\). Thus, this third-degree polynomial equation has three roots, which verifies the corollary above. The roots of a polynomial equation may be different real numbers or they may be complex. For example, \( x^3 + x = 0 \) has three roots: \(0, i, \) and \(-i\). You should verify this by factoring.

Many problems can be solved by using any one of a number of different strategies. Sometimes it takes more than one strategy to solve a problem.

**Example**

Find all roots of \( 0 = x^3 + 3x^2 - 10x - 24 \).

We can find the roots of the equation by combining some of the strategies that you have previously learned. Let's **list some possibilities** for roots and then eliminate those that are not roots. Suppose we begin with integral values from \(-4\) to \(4\) and use the shortened form of synthetic substitution. Because this polynomial function has degree 3, the equation has three roots. However, some of them may be imaginary.

The related function of the equation is \( f(x) = x^3 + 3x^2 - 10x - 24 \). You can use either synthetic substitution or a scientific calculator to find \( f(a) \) quickly. Find \( f(-4) \).

**Method 1: Synthetic Substitution**

\[
\begin{array}{cccc}
-4 & 1 & 3 & -10 \\
& -4 & 4 & 24 \\
1 & -1 & -6 & 0 \\
\end{array}
\]

(continued on the next page)
Method 2: Scientific Calculator

Enter: \[4 \pm 3 \Rightarrow -i\]
\[\times \quad \text{RCL} \quad + \quad 10 \pm \Rightarrow -5\]
\[\times \quad \text{RCL} \quad + \quad 24 \pm \Rightarrow 0\]

The display shown after each \(\Rightarrow\) gives the second, third, and fourth coefficients of the depressed polynomial. To evaluate \(f(x)\) for other values for \(x\), simply change the first number entered in the series of keystrokes shown above.

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>3</th>
<th>-10</th>
<th>-24</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>1</td>
<td>-1</td>
<td>-6</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>1</td>
<td>0</td>
<td>-10</td>
<td>6</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>-12</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>-12</td>
<td>-12</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-10</td>
<td>-24</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>-6</td>
<td>-30</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>-24</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

The zeros occur at \(x = -4, x = -2,\) and \(x = 3\). The graph of the function verifies that there are three real roots.

Remember when you solved a quadratic equation like \(x^2 + 9 = 0\), there were always two imaginary roots. In this case, \(3i\) and \(-3i\) are the roots. These numbers are a conjugate pair. In any polynomial function, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the complex conjugates theorem.

**Complex Conjugates Theorem**

Suppose \(a\) and \(b\) are real numbers with \(b \neq 0\). If \(a + bi\) is a zero of a polynomial function, then \(a - bi\) is also a zero of the function.

**Example 2**

Find all zeros of \(f(x) = x^3 - 5x^2 - 7x + 51\) if \(4 - i\) is one zero of \(f(x)\).

Since \(4 - i\) is a zero, \(4 + i\) is also a zero, according to the complex conjugates theorem. So, both \(x - (4 - i)\) and \(x - (4 + i)\) are factors of the polynomial \(x^3 - 5x^2 - 7x - 51\).

\[
f(x) = [x - (4 - i)][x - (4 + i)](-i)^2
\]
\[
= [x^2 - (4 + i)x - (4 - i)x + (4 - i)(4 + i)](-i)
\]
\[
= (x^2 - 4x - ix - 4x + ix + 16 - i^2)(-i)
\]
\[
= (x^2 - 8x + 17)(-i)
\]

Remember that \(-i^2 = 1\).

Since \(f(x)\) has degree 3, there are three factors. Use division to find the third factor.
\[ x^2 - 8x + 17x^3 - 5x^2 - 7x + 51 \]
\[ x^3 - 8x^2 + 17x \]
\[ 3x^2 - 24x + 51 \]
\[ 3x^2 - 24x + 51 \]
Subtract.

Therefore, \( f(x) = (x^2 - 8x + 17)(x + 3) \). Since \( x + 3 \) is also a factor, \(-3\) is also a zero. The three zeros are \( 4 - i \), \( 4 + i \), and \(-3\). The graph verifies the nature of the zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

**Descartes' Rule of Signs**

If \( P(x) \) is a polynomial function whose terms are arranged in descending powers of the variable,

- the number of positive real zeros of \( P(x) \) is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of \( P(x) \) is the same as the number of changes in sign of the coefficients of the terms of \( P(-x) \), or is less than this by an even number.

**Example**

State the number of positive and negative real zeros for \( p(x) = 4x^5 + 3x^4 - 2x^3 + 5x^2 - 6x + 1 \).

Use Descartes' rule of signs. Count the number of changes in sign for the coefficients of \( p(x) \).

\[ p(x) = 4x^5 + 3x^4 - 2x^3 + 5x^2 - 6x + 1 \]

\[
\begin{array}{cccccc}
4 & 3 & -2 & 5 & -6 & 1 \\
\text{no} & \text{yes} & \text{yes} & \text{yes} & \text{yes} \\
\end{array}
\]

Since there are four sign changes, there are either 4, 2, or 0 positive real zeros.

Find \( p(-x) \) and count the number of changes in signs for its coefficients.

\[ p(-x) = 4(-x)^5 + 3(-x)^4 - 2(-x)^3 + 5(-x)^2 - 6(-x) + 1 \]
\[ = -4x^5 + 3x^4 + 2x^3 + 5x^2 + 6x + 1 \]

\[
\begin{array}{cccccc}
\text{yes} & \text{no} & \text{no} & \text{no} & \text{no} \\
\end{array}
\]

Since there is one sign change, there is exactly 1 negative real zero.

Thus, the function \( p(x) \) has either 4, 2, or 0 positive real zeros and exactly 1 negative real zero.
Using a graphing calculator or sketching the graph may help in determining the nature of the zeros of a function.

In Example 3, since \( p(x) \) has degree 5, it has five zeros. Using the information in the example, you can make a chart of the possible combinations of real and imaginary zeros.

<table>
<thead>
<tr>
<th>Number of Positive Real Zeros</th>
<th>Number of Negative Real Zeros</th>
<th>Number of Imaginary Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Example 4

Write the polynomial function of least degree with integral coefficients whose zeros include 7 and \( 3 + 2i \).

If \( 3 + 2i \) is a zero, then \( 3 - 2i \) is also a zero. Why?

Use the zero product property to write a polynomial equation that has these zeros, \( 7, 3 + 2i, \) and \( 3 - 2i \), as roots.

\[
0 = (x - 7)(x - (3 + 2i))(x - (3 - 2i))
\]

So,

\[
f(x) = (x - 7)(x - (3 + 2i))(x - (3 - 2i))
= (x - 7)(x^2 - (3 - 2i)x - (3 + 2i)x + (3 + 2i)(3 - 2i))
= (x - 7)(x^2 - 3x + 2ix - 3x - 2ix + 9 - 4i^2)
= (x - 7)(x^2 - 6x + 13)
= x^3 - 13x^2 + 55x - 91
\]

**CHECK FOR UNDERSTANDING**

**Communicating Mathematics**

Study the lesson. Then complete the following.

1. Write an example of each function. List the possibilities for its zeros.
   a. quadratic  
   b. cubic  
   c. quartic
2. Describe the complex conjugate theorem. Use \( 6 + 7i \) as an example.
3. State the zeros of a polynomial function if \((x + 6)\) and \([x - (5 + i)]\) are factors of the polynomial. \(-6, 5 + i, 5 - i\)
4. a. Write a polynomial function \( p(x) \) whose coefficients have four sign changes.  
   b. Find the number of sign changes that \( p(-x) \) has.
   c. Describe the nature of the zeros.
5. The graph of \( f(x) = x^3 - 8 \) is shown at the right.
   a. Describe the nature of the zeros.
   b. Find the zeros of the function.
6. A classmate has been out ill and missed learning Descartes' rule of signs. Write an example and explain Descartes' rule of signs to your fellow classmate. See students' work.
7. \( f(-x) = 6x^4 + 3x^3 + 5x^2 + x + 2 \)  
   a. Find \( f(-x) \) for each function given.
   b. \( f(-x) = -x^7 + x^3 - 2x - 1 \)

**Guided Practice**

7. \( f(x) = 6x^4 - 3x^3 + 5x^2 - x + 2 \)  
8. \( f(x) = x^7 - x^3 + 2x - 1 \)
State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

9. \(f(x) = x^3 - 6x^2 + 1\)  
10. \(f(x) = x^4 + 5x^3 + 2x^2 - 7x - 9\)
   2 or 0; 1 or 2  
   1; 3 or 1; 0 or 2

Given a function and one of its zeros, find all of the zeros of the function.

11. \(h(x) = x^3 - 6x^2 + 10x - 8; 4\)  
12. \(g(x) = x^3 + 6x^2 + 21x + 26; -2\)
13. \(f(x) = x^3 + 7x^2 + 25x + 175; 5\)  
14. \(p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40; 3 - i\)  

Write the polynomial function of least degree with integral coefficients that has the given zeros.

15. \(-4, 1, 5\)  
16. \(9, 1 + 2i\)

17. Manufacturing The volume of a candy carton is 120 in\(^3\). To hold the correct number of candy bars, the carton must be 3 inches longer than it is wide. The height is 2 inches less than the width. Find the dimensions of the carton. \(l = 8 \text{ in.}, w = 5 \text{ in.}, h = 3 \text{ in.}\)

**EXERCISES**

**Practice**

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function. 18–25. See margin.

18. \(f(x) = 5x^3 + 8x^2 - 4x + 3\)  
19. \(g(x) = x^4 + x^3 + 2x^2 - 3x - 1\)
20. \(h(x) = 4x^3 - 6x^2 + 8x - 5\)  
21. \(f(x) = x^4 - 9\)
22. \(r(x) = x^5 - x^3 - x + 1\)  
23. \(g(x) = x^{14} + x^{10} - x^8 + x - 1\)
24. \(p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1\)  
25. \(f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1\)

Given a function and one of its zeros, find all of the zeros of the function.

26. \(p(x) = x^3 + 2x^2 - 3x + 20; -4\)  
27. \(f(x) = x^3 - 4x^2 + 6x - 4; 2\)
28. \(v(x) = x^3 - 3x^2 + 4x - 12; 2i\)  
29. \(h(x) = 4x^4 + 17x^2 + 4; 2i\)
30. \(g(x) = 2x^3 - x^2 + 28x + 51; -\frac{3}{2}\)  
31. \(q(x) = 2x^3 - 17x^2 + 90x - 41; \frac{1}{2}\)
32. \(f(x) = x^3 - 3x^2 + 9x + 13; 2 + 3i, 2 - 3i, -1\)
33. \(r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13; 3 + 2i, 3 - 2i, 3 + 2i, -1, 1\)
34. \(h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156; 5 - i, 5 + i, -1, 6\)

Write the polynomial function of least degree with integral coefficients that has the given zeros. 35–40. See margin.

35. \(-2, 1, 3\)  
36. \(2, 4i\)
37. \(4i, 3, -3\)
38. \(3, 1 + i\)  
39. \(2i, 3i, 1\)
40. \(6, 2 + 2i\)

41. If \(f(x) = x^3 + hx^2 - 7x - 15\), find the value of \(h\) so that \(-2 - i\) is a zero of \(f(x)\).

42. Suppose a fifth-degree polynomial has exactly two \(x\)-intercepts. Describe the nature of the roots of the function. Sketch some examples to support your reasoning. See Solutions Manual.
43. Medicine  Doctors can measure cardiac output in patients at high risk for a heart attack by monitoring the concentration of dye injected into a vein near the heart. A normal heart’s dye concentration is approximated by 
\[ d(t) = -0.006x^4 - 0.140x^3 - 0.053x^2 + 1.79x \]
where \( x \) is the time in seconds.

(a) Find all real zeros by graphing. Then verify them by using synthetic division. \(-22.3, -4.2, 0, 3.2\); See margin for graph.

(b) Which root makes sense for an answer to this problem? Why? \(3.2\)

44. Physiology  During a five-second respiratory cycle, the volume of air in liters in the human lungs can be described by the function \( A(t) = 0.1729t^3 + 0.1522t^2 - 0.0374t^3 \), where \( t \) is the time in seconds. Find the volume of air held by the lungs at 3 seconds. \( 0.8787 \) liter

45. Graph \( f(x) = x^3 - 5x + 7 \). \( \text{(Lesson 8-3)} \) See Solutions Manual.

46. Find the center and radius of a circle whose equation is \( x^2 + (y - 3)^2 - 4x - 77 = 0 \). \( \text{(Lesson 7-3)} \) \((2,3); 9\)

47. Write a quadratic equation that has roots 3 and \(-5\). \( \text{(Lesson 4-2)} \)

48. Find
\[
\begin{bmatrix}
-2 & 5 \\
4 & 9 \\
\end{bmatrix} + \begin{bmatrix}
5 & 4 \\
9 & 8 \\
\end{bmatrix} = \begin{bmatrix}
3 & 10 \\
14 & 17 \\
\end{bmatrix}
\]

49. Design  Marco is designing a new dartboard. The center of the board is defined by the inequality \( |x| + |y| \leq 2 \). Draw the graph of this inequality to see what Marco’s new dartboard will look like. \( \text{(Lesson 3-4)} \)

50. Name which ordered pairs, \((7, -3),( -4, -1)\), or \((12, -6)\), satisfy \(-2|x| - 5y < 3\). \( \text{(Lesson 2-7)} \)

51. Find the value of \( f(12) \) when \( f(x) = \frac{19}{23 - x} \). \( \text{(Lesson 2-1)} \) \(19\)

52. Evaluate \(-4| -5x | + 17 \) if \( x = 2 \). \( \text{(Lesson 1-5)} \) \(-23\)

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**SELF TEST**

Find each value if \( p(x) = 4x^3 - 3x^2 + 2x - 5 \). \( \text{(Lesson 8-1)} \)

1. \( p(a^2) \)
2. \( p(x + 1) \)

\( \text{Given a polynomial and one of its factors, find the remaining factors of the polynomial.} \) \( \text{(Lesson 8-2)} \)

3. \( x^3 + x^2 - 24x + 36; x - 3, x + 6, x - 2 \)
4. \( 2x^3 + 13x^2 + x - 70; x - 2, 2x + 7, x + 5 \)

Graph each function. \( \text{(Lesson 8-3)} \) \( \text{5-6. See margin.} \)

5. \( g(x) = x^3 - 5 \)
6. \( h(x) = x^3 - x^2 + 4 \)

**State the number of positive real zeros, negative real zeros, and imaginary zeros for each function. \( \text{(Lesson 8-4)} \)

7. \( f(x) = x^3 + 8x^2 - 7x + 10 \)
8. \( f(x) = 6x^4 + 18x^3 + 4x - 9 \)

9. Determine whether the degree of the function represented by the graph at the right is even or odd. How many real zeros does the polynomial function have? \( \text{(Lesson 8-1)} \) \text{even, 4} \)

10. Manufacturing  The height of a certain juice can is 4 times the radius of the top of the can. Determine the dimensions of the can if the volume is approximately 17.89 cubic inches. \( \text{(Hint: The formula for the volume of a right circular cylinder is} \ V = \pi r^2 h) \) \( \text{(Lesson 8-4)} \) \( r \approx 1.125 \text{ in., } h \approx 4.5 \text{ in.} \)

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508  Chapter 8  Exploring Polynomial Functions
Rational Zero Theorem

What You'll Learn

- To identify all possible rational zeros of a polynomial function by using the rational zero theorem, and
- To find zeros of polynomial functions.

Why It's Important

You can use the rational zero theorem to find zeros of polynomials that model situations in finance and food production.

Application

Architecture

The largest pyramid in the United States is the Luxor Hotel and Casino in Las Vegas, Nevada. The volume of this unique hotel and casino is 28,933,800, or about 29 million cubic feet. The height of the pyramid is 148 feet less than the length of the building. The base of the building is square. What are the dimensions of this building?

The formula for the volume of a pyramid is $V = \frac{1}{3}Bh$, where $B$ represents the area of the base and $h$ represents the height. Let's set up an equation to find the dimensions of the pyramid. Let $s$ represent the length of one side of the base of the pyramid. Then the height is $s - 148$.

\[
\begin{align*}
V &= \frac{1}{3}Bh \\
28,933,800 &= \frac{1}{3}(s^2)(s - 148) \\
86,801,400 &= s^2(s - 148) \\
86,801,400 &= s^3 - 148s^2 \\
0 &= s^3 - 148s^2 - 86,801,400
\end{align*}
\]

We could use synthetic substitution to test possible zeros. But the numbers are so large that we might have to test hundreds of possible zeros before we find one. In situations like this, the rational zero theorem can give us some direction in testing possible zeros. This theorem and a corollary are stated below.

<table>
<thead>
<tr>
<th>Rational Zero Theorem</th>
<th>Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then $p$ is a factor of $a_n$ and $q$ is a factor of $a_0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary (Integral Zero Theorem)</td>
<td>If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of $a_n$.</td>
</tr>
</tbody>
</table>