Conceptualizing Mathematically Significant Pedagogical Openings to Build on Student Thinking

Keith R. Leatham and Blake E. Peterson
Brigham Young University

Shari L. Stockero
Michigan Technological University

Laura R. Van Zoest
Western Michigan University

Correspondence concerning this article should be addressed to Keith R. Leatham, 193A TMCB, Department of Mathematics Education, Brigham Young University, Provo, UT 84602; kleatham@mathed.byu.edu
Abstract

The mathematics education community values using student thinking to develop mathematical concepts, but the nuances of this practice are not clearly understood. We conceptualize an important group of instances in classroom lessons that occur at the intersection of student thinking, significant mathematics, and pedagogical openings—what we call Mathematically Significant Pedagogical Openings to Build on Student Thinking (MOSTs). We analyze dialogue to illustrate the process for determining whether a classroom instance is a MOST. We conclude by discussing how the MOST construct contributes to facilitating and researching teachers' mathematically-productive use of student thinking through providing a lens and generating a common language for recognizing and agreeing upon high-leverage student mathematical thinking.

*Keywords:* student thinking, teaching practice, classroom mathematics discourse, professional development, teachable moments
Conceptualizing Mathematically Significant Pedagogical Openings to Build on Student Thinking

Research in mathematics teacher education suggests the benefits of instructional practices that build on student thinking (e.g., Fennema, et al., 1996; Stein & Lane, 1996), but such practices are complex and difficult both to understand and to enact (Ball & Cohen, 1999; Feiman-Nemser, 2001; Sherin, 2002; Silver, Ghouseiini, Gosen, Charalambous, & Font Strawhun, 2005). Often opportunities to use student thinking to further mathematical understanding either go unnoticed or are not acted upon by teachers, particularly novices (Peterson & Leatham, 2009; Stockero & Van Zoest, 2012). Despite a growing number of teachers who are convinced of the value of student thinking and the need to encourage it, neither teachers nor those who educate them have a clear understanding of what thinking can best be used to develop mathematical concepts (Peterson & Leatham, 2009; Van Zoest, Stockero, & Kratky, 2010). We address this issue by providing a conceptual framework for thinking about the instances of student mathematical thinking that emerge while teaching. We refer to high-leverage instances of student thinking—those that have the most potential to increase student understanding of important mathematical ideas—as Mathematically Significant Pedagogical Openings to build on Student Thinking (MOSTs).

One difficulty in learning to use students’ mathematical thinking is related to the complexity of engaging in, interpreting and facilitating classroom mathematics discourse. As Lewis (2008) observed, “The ‘real’ classroom experience is elusive: each moment is experienced differently by the actors involved and their perceptions of those experiences change with time and reflection. The choices of what to focus on, which story to follow, are endless” (p. 5). Although we acknowledge that people will indeed experience classroom events differently, we are nevertheless convinced that there are key elements of classroom interactions that can be
recognized and agreed upon by teachers and teacher educators as being supportive of student learning. One critical role that mathematics teacher education can play is to provide lenses, informed by research and advocated by the community at large, to facilitate this mutual recognition and agreement. The conceptual framework put forth in this paper is designed to be such a lens.

Our work is part of ongoing efforts to understand the practice of eliciting and productively using students’ mathematical thinking during classroom instruction. Stein, Smith and colleagues’ work on the cognitive demand of tasks (e.g., Stein, Smith, Henningsen, & Silver, 2009) has been instrumental in supporting the elicitation of student mathematical thinking in classrooms and is, in part, responsible for generating the need to know how to respond to that thinking once it has surfaced. Their work has also focused on orchestrating productive classroom discussion around high-cognitive-demand tasks. Partly in response to earlier findings about the tendency for cognitive demand to degrade during enactment of high quality tasks (e.g., Stein, Grover, & Henningsen, 1996), Smith and Stein (2011) articulated five practices focused on orchestrating classroom discussions around student solutions to tasks. This focus on discussing student work in response to a task can be seen as a “subpractice” of the practice of using student thinking productively. That is, it is using a specific kind of student thinking—that which emerges when doing a mathematical task—as the basis for a whole-class discussion of the mathematical ideas that the task is designed to elicit. Although identifying and capitalizing on MOSTs does provide a means to maintain the cognitive demand of tasks, the MOST construct is designed both to take in a broader range of student actions (including, for example, student actions in response to a lecture or student comments that emerge during a discussion of homework) and to provide a mechanism for determining which particular instantiations of these actions provide opportunities
to productively use student mathematical thinking to leverage learning. For example, the MOST construct is consistent with Smith and Stein’s (2011) views on selection: “The selection of particular students and their solutions is guided by the mathematical goal for the lesson and the teacher’s assessment of how each contribution will contribute to that goal” (p. 10), but goes into depth about these connections to mathematical goals and whether a particular student solution provides a pedagogical opening to work toward those goals. In addition, Smith and Stein focused much of their work on anticipating different solutions, whereas the MOST construct provides teachers a mechanism to evaluate all instances of student thinking—both anticipated and unanticipated.

Our work is also closely connected to the growing body of mathematics education research focused on teacher noticing (e.g. Sherin, Jacobs, & Philipp, 2011). Sherin and van Es (e.g., Sherin & van Es, 2005; van Es & Sherin, 2002) define noticing as comprised of three interrelated skills: identifying important events during instruction, reasoning about them, and making connections between the events and broader educational principles. The MOST construct focuses primarily on developing the first two skills, as we see it as a tool for recognizing and analyzing classroom instances that have the potential to increase student understanding of important mathematical ideas. The construct also contributes to work on teacher noticing by clearly defining “important events” in a mathematics classroom.

We focus attention on the MOST construct because of its potential to contribute to the work of facilitating and researching teachers’ mathematically-productive use of student thinking. Although this paper is about characterizing and recognizing MOSTs (as opposed to using them), they are better understood if the reader has a sense of our vision of how they might be productively used to support student learning. A teacher may respond to MOSTs in a variety of
different ways, from inserting a teacher explanation to asking follow-up questions to orchestrating a class discussion. When a teacher sees a MOST as an opportunity to step in and explain, it could be classified as naïve use (Peterson & Leatham, 2009) in that the teacher may be using the MOST merely as a trigger to lecture about the mathematical topic, rather than to build on the student thinking. A more productive use of a MOST is to orchestrate a discussion around the mathematics at hand. This orchestration could be done, for example, by posing questions that focus the class on connections between the mathematics of the observed student thinking and other concepts that are related to the mathematical goals of the classroom.

The MOST construct contributes to research on productive use of student mathematical thinking primarily through providing a lens and generating a common language for recognizing and agreeing upon high-leverage instances of student mathematical thinking. Specifically, it contributes to the work of facilitating teacher learning by providing guidance for identifying the aspects of students’ mathematical thinking that are most productive to focus on in preservice teacher coursework and inservice teacher professional development. It also provides a framework and language for conversation among teacher educators and teachers about high-leverage student thinking. Similarly, it contributes to researching teachers’ use of student thinking by providing a lens to focus classroom discourse analysis on student mathematical thinking and tools to assess which student mathematical thinking is high-leverage. In this paper we describe a conceptual framework centered on the MOST construct. We then analyze classroom episodes from the literature in order to elaborate on the MOST construct and to illustrate the analysis process for determining whether a classroom instance is a MOST.
Mathematically Significant Pedagogical Openings to Build on Student Thinking

Although skilled teachers and teacher educators often recognize when important mathematical moments occur during a lesson and can readily produce ideas about how to capitalize on them, the literature reveals a construct that is neither well-defined nor explicitly articulated. While not the focus of extant literature, such instances are mentioned in a number of different ways. For example, Jaworski (1994) referred to “critical moments in the classroom when students created a moment of choice or opportunity” (p. 527). Davies and Walker (2005) used the term “significant mathematical instances” (p. 275), Davis (1997) used “potentially powerful learning opportunities” (p. 360) and Thames and Ball (in press) used “crucial mathematic hinge moment[s]” (p. 26). Schoenfeld (2008) referred to moments that contained “the fodder for a content-related conversation” (p. 57), “an issue that the teacher judges to be a candidate for classroom discussion” (p. 65) and the “grist for later discussion or reflection” (p. 70). Schifter (1996) spoke of “novel student idea[s] that prompt teachers to reflect on and rethink their instruction” (p. 130).

It is clear from the literature that these instances, whatever they are called, are important to mathematics teaching and learning. In studying such references to these instances and drawing on our own classroom and research experiences, we have identified three critical characteristics of these moments: student thinking, significant mathematics, and pedagogical openings. In the following sections we define these three characteristics. We then locate MOSTs — Mathematically Significant Pedagogical Openings to Build on Student Thinking—in the intersection of these three characteristics.
Mathematically Significant Pedagogical Openings To Build On Student Thinking

Student Thinking

Because the MOST construct is designed to help articulate productive use of student mathematical thinking, we begin by defining what we mean by student thinking. We recognize our inability to access directly the thoughts of students. Instead we make inferences based on our observations of what they say and do. Teachers (and researchers) must “listen to the student, interpret what the student does and says, and try to build a ‘model’ of the student’s conceptual structures” (von Glasersfeld, 1995, p. 14). Thus, when we use the phrase student thinking we refer to observable evidence of student thinking, which we define as any instance where a student’s actions provide sufficient evidence to make reasonable inferences about their thinking. In the classroom setting, this evidence most commonly is visible in verbal utterances, board-work, or gestures that occur in whole-class discussions, but it may also be observable in small-group settings or individual written work.

Note that we make a distinction between observable and observed. There are many cases, particularly with novice teachers, where student thinking is observable, but not observed by the teacher (e.g., Berliner, 2001; Peterson & Leatham, 2009; Stockero & Van Zoest, 2012). One explanation for this phenomenon is inattentional blindness (Simons, 2000)—described in the psychology literature as a failure to focus attention on unexpected events. In addition, these ideas are closely tied to teacher noticing (e.g., Sherin, et al., 2011)—what a teacher attends to (or fails to attend to) during a lesson. In the context of teaching, the teacher’s failure to observe student thinking may mean that the teacher is not paying attention to student thinking or does not notice a particular instance of student thinking, rather than that there is no observable evidence of student thinking. Thus, for the purposes of our work, observable refers to thinking that could be observed by someone (e.g., the teacher, other students, a researcher) who witnessed the instance,
either by being present or by engaging with a record of the interactions.

We also make the distinction between evidence that students are thinking and evidence of what students might be thinking. Although simple yes/no answers or other utterances may provide evidence that students are thinking, they generally do not provide insight into what students are thinking about mathematical ideas, and thus would not provide sufficient evidence to make reasonable inferences about their thinking.

In conclusion, to determine whether an instance involves student thinking, we focus on student actions that allow us to make reasonable inferences about their thinking. One of the advantages of this focus is that it shifts attention away from the teacher. This is not to say that teacher actions are not relevant—to the contrary, they can create an environment supportive of student thinking—but that only student actions can generate a MOST.

Mathematically Significant

In order to be a MOST, the mathematics in an instance must warrant use of limited instructional time; that is, it must be what we call mathematically significant. We use the term mathematically significant in the context of teachers engaging a particular group of students in the learning of mathematics. Thus, we see it as a subset of important mathematics, which can be determined apart from a specific classroom context. In the mathematical analysis of an instance, we consider mathematically significant in relationship to three key criteria: the importance of the mathematical idea of the instance, the appropriateness of the mathematics to the students in the classroom, and the extent to which the mathematics is connected to the mathematical goals for this group of students.

Important mathematics. The first criterion for a mathematically significant instance is that there is mathematics within the instance that is related to an important mathematical idea. To
determine whether this criterion is met one must first determine whether the student thinking is mathematical in nature and, if so, what mathematics the student is expressing—what we call the mathematics of the instance. If the mathematics of the instance cannot be determined without gathering more information, the instance, by default, would not meet the important mathematics criterion. The instance may, however, prompt a teacher move to gather more information. If this move were effective in surfacing an important mathematical idea, a related instance later in the discussion might meet the important mathematics criterion.

In order to determine the importance of the mathematics of the instance, one must be able to articulate an important mathematical idea that is closely related to the mathematics of the instance. Because this determination is purely mathematical, it can be made independent of a particular classroom context. Curriculum guidance documents such as the Common Core State Standards for Mathematics [CCSS-M] (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA-CCSSO], 2010) and the Principles and Standards for School Mathematics [PSSM] (National Council of Teachers of Mathematics [NCTM], 2000) articulate important mathematical ideas for all students. Also important to the study of mathematics are mathematical practices, such as those outlined in the CCSS-M (pp. 6-8): (a) make sense of problems and persevere in solving them; (b) reason abstractly and quantitatively; (c) construct viable arguments and critique the reasoning of others; (d) model with mathematics; (e) use appropriate tools strategically; (f) attend to precision; (g) look for and make use of structure; and (h) look for and express regularity in repeated reasoning. These mathematical practices are important across the learning spectrum because they get at the heart of the discipline. Therefore, in the context of determining whether an instance is mathematically
significant, mathematical practices—when there is strong evidence that students are engaging in them—are mathematically important by their very nature.

**Appropriate mathematics.** A second criterion for mathematically significant is that the mathematics of the instance be appropriate for the students in the classroom. That is, it must help students develop mathematically and move forward in their learning. Meeting this criterion requires two things. First, the mathematics of the instance must be accessible to the students given their prior mathematical experiences; they must have adequate background knowledge to engage with the mathematical idea. Second, the students must not yet have mastered the mathematical idea related to the mathematics of the instance. If they had, pursuing that idea would not likely move them forward in their learning. Thus, the appropriate mathematics criterion requires that the mathematical idea be accessible to students with a particular level of mathematical experience while not being likely to have been already mastered. Documents such as the CCSS-M (NGA-CCSSO, 2010) and the PSSM (NCTM, 2000), as well as research on learning trajectories (e.g., Clements & Sarama, 2009; Confrey, Maloney, & Nguyen, in press; Steffe, 2004; Stylianides, 2008), provide guidelines for background knowledge needed to understand mathematical ideas that can be helpful in determining whether the mathematics of an instance is appropriate for students with particular mathematical experiences.

To illustrate the appropriate mathematics criterion, consider the important mathematical idea of one-to-one correspondence. When children are learning to count, the idea that their uttered cadence of “one, two, three, …” must have a one-to-one correspondence with the objects at which they are pointing is mathematically appropriate for students at that point in their mathematical development. An instance in which this same idea surfaced with more mathematically advanced children would not help them move forward in their learning because
they would likely have already mastered the idea, so would not meet the appropriate mathematics criterion.

As a second example, consider the important mathematical idea of finding the area of an irregularly-shaped object. In calculus this idea is related to finding the area under a curve using a limiting process with rectangles, while in earlier grades it is related to the idea that the area of any object that is composed of simple shapes can be found by finding the sum of the areas of those shapes. The idea that a limit can be used to accurately find the area under a curve would be mathematically appropriate in a calculus course where limits are being studied, even if limits had not yet been used in this way. If this same idea were to arise in an earlier grade, however, it would not be mathematically appropriate because the students likely would not have the background knowledge to make sense of it. In the latter case, even though the instance involves important mathematics, it would fail the appropriate mathematics criterion and thus would not be mathematically significant.

**Mathematical goals.** A third criterion of mathematically significant is that there is a clear mathematical connection between the mathematical idea related to the instance and mathematical goals for student learning in that class. The mathematical goals for the classroom encompass both mathematical content and mathematical practices. They could be determined by the teacher or by an external source, such as curriculum documents, or they could be inferred by an observer who is knowledgeable in the field of mathematics education, such as a researcher or teacher educator. When analyzing the mathematical idea related to an instance in relation to the mathematical goals for student learning, it is important to consider a range of goals, from those for the lesson in which the instance occurs, to those for the unit of instruction in which the lesson occurs, for the course students are taking, or for their broader mathematical learning (see Figure
1). In the case of lesson goals, the instance may focus on a particular mathematical idea or connections among ideas within a lesson. In the other cases, the instance might involve making connections to other areas of mathematics, revisiting ideas from prior courses, or previewing ideas from future courses. Developing mathematical ways of thinking, such as the mathematical practices listed in the CCSS-M (NGA-CCSSO, 2010), could be goals at any of these levels.

---

**Figure 1. Layers of mathematical goals to which a MOST may relate.**

Figure 1 depicts the range of and relation among goals for student learning. The figure contains concentric circles, with each circle embedded within those around it: lesson goals build to meet unit goals, unit goals build to meet course goals, and course goals help to meet broader goals for students’ mathematical learning. Each ring represents goals that are not explicitly in the smaller embedded circle(s). For example, the unit goal ring indicates goals that are not intended to be addressed in the lesson under consideration, but that are explicit goals for student learning in the short-term. Similarly, the broader mathematical goals ring indicates goals that are not explicit in a given course, unit or lesson, but that are important to students’ overall mathematical learning.
Determining whether the mathematical goals criterion has been met—whether there is a clear mathematical connection between the mathematics of the instance and mathematical goals for student learning in that class—relates to where in Figure 1 the goal lies. Important and appropriate mathematics directly connected to a lesson goal, even if the connection is weak, will meet the criterion because of the immediacy of the goal. Instances related to goals further from the center of the figure require stronger connections. In addition, these goals must be high-priority goals for students’ overall mathematical understanding to warrant use of limited instructional time during a lesson in which they are not an explicit instructional goal. In short, the threshold for meeting the mathematical goals criterion is related to the proximity of the goal to the lesson.

**Determining whether an instance is mathematically significant.** We now explain how we analyze the mathematics of an instance with respect to all three criteria in order to determine whether an instance is mathematically significant. In the process, we illustrate the range of ways the mathematics of an instance could be mathematically connected to mathematical goals for students in a classroom.

The first step in engaging in a mathematical analysis of a classroom instance is to determine the mathematics of the instance. Recall that if this mathematics cannot be determined based on the information available in the instance, the instance cannot be mathematically important and thus is not mathematically significant. When the mathematics of the instance can be determined, we see the assessment of the three individual criteria for mathematically significant—important mathematics, appropriate mathematics and mathematical goals—occurring linearly. Figure 2 depicts the analysis process and provides questions that encapsulate each criterion. As illustrated in the figure, mathematical importance is determined first, followed
by mathematical appropriateness, and then connectedness to mathematical goals. If any criterion is not met, the analysis ends, and the instance is judged not to be mathematically significant.

**Figure 2.** Criteria to determine whether the mathematics of an instance is mathematically significant.

Although there are non-mathematical goals for students’ learning in a classroom, when considering whether an instance is mathematically significant using this approach, the focus is automatically narrowed to goals for students’ learning that are distinctly mathematical. For example, helping students work more productively in small groups is a goal in many classrooms. Although this goal may support students’ mathematical learning, since the goal itself is not mathematical in nature, an instance related only to this goal would not make it past the important mathematics criterion, and therefore would not be mathematically significant. Likewise, a student comment that summarizes an important mathematical idea that is already well-understood by the class as a whole would meet the important mathematics criterion, but would fail to meet the appropriate mathematics criterion, so would not be mathematically significant. To be clear, we are not saying that instances such as these cannot be important to a lesson, but instead that pursuing these instances does not contribute directly to furthering students’ mathematical understanding. For example, if the aforementioned student (who summarized a mathematical idea that was already well-understood by the class) had never engaged in public discussion about mathematics before, the teacher may well chose to pursue the instance because
it provided a pathway to an important pedagogical goal for the class—involving all students in class discussion. The power of the mathematically significant construct is that it provides a mechanism for determining which instances will help to meet the mathematical learning goals for the students in a classroom.

We now illustrate the process of analyzing the mathematics of an instance and, in particular, the range of ways that the related important mathematical idea could be connected to mathematical goals for student learning. Consider a beginning algebra lesson where the goal is for students to understand the relationship between the graph of a linear function and its equation written in slope-intercept form, \( y = mx + b \). A student question about whether the \( y \)-intercept of the graph is always the same as the \( b \) value in the equation involves important and appropriate mathematics that is at the heart of the mathematical goal of the lesson, so an instance in which this question arises clearly meets all three criteria, and thus is mathematically significant. A student question related to whether there could be two \( y \)-intercepts involves an important mathematical idea that is more distant from the lesson goal, but is appropriate for this group of students because it offers the opportunity for students to better understand the definition of function. Because understanding the definition of function is likely an important learning goal for the beginning algebra course, the instance in which this question arises also meets the mathematical goals criterion, and thus is mathematically significant.

Now suppose the mathematics department of this school had identified the goal that all students in their program would develop the mathematical practice of making use of structure (NGA-CCSSO, 2010). In this context, consider a student question about whether the \( y \)-intercept of the graph of any function is the same as the constant in the equation. The important mathematical concept underlying this question—that this generalization is not true for many non-
Mathematically Significant Pedagogical Openings To Build On Student Thinking

Polynomial functions—may be a learning goal for the students beyond the course, but is not likely appropriate for them, as they would not have the mathematical background required to understand the idea. Typically, students experiencing this lesson would not yet have studied other families of functions, so pursuing this idea would detract from, rather than help meet, the mathematical goals for the students in the class. However, the student’s question provides evidence of engagement in the mathematical practice of looking for and making use of structure. This is an important mathematical practice that is appropriate for students at this level. Because developing this practice is a programmatic goal for these students, this instance is directly connected to goals in the outer ring of Figure 1. Therefore, even though the mathematical concept underlying this idea does not satisfy all three criteria of mathematically significant, the mathematical practice does. This example illustrates how an instance can involve more than one important mathematical idea, in this case, both a mathematical concept and a mathematical practice. It also shows how even when the important mathematical concept in an instance may not be appropriate for a particular group of students, an important mathematical practice within it may be appropriate and provide a mathematical connection to mathematical goals for the students’ learning.

In considering the relationship of the mathematics of instances to goals for student learning, we have come to understand mathematical practices as having two components: the practice as a topic—something one learns about—and the practice as something one does. We hypothesize that the practice as a topic, by the nature of mathematical practices, is consistently important, appropriate, and provides a mathematical connection to meeting goals in the outer ring of Figure 1 (as well as possibly those in the inner circles), while the practice as something one does is more contextual. For example, an instance in which a student questions whether his
argument is viable would provide a direct connection to the goal of developing the mathematical practice of constructing viable arguments and critiquing the reasoning of others—a practice central to the discipline of mathematics and, arguably, a broad goal for all mathematics learners.

In this case, the practice itself could become a topic of classroom discussion. On the other hand, a student engaging in the practice of constructing a mathematical argument is contextual, as whether it is appropriate mathematics for a particular group of students depends on their prior mathematical experiences. A student articulating clearly what they know and what they are trying to prove would be appropriate mathematics to discuss when the class is first learning to construct proofs, but likely would not be once this practice had become routine.

**Summary.** We have defined an instance in a given classroom to be mathematically significant if it meets the three criteria of important mathematics, appropriate mathematics, and connections to mathematical goals for student learning. Although the first criterion can be assessed without knowledge of the context in which an instance arises, the latter two cannot. Therefore it is not possible to determine whether an instance is mathematically significant without taking the classroom context into consideration. A mathematical topic may be significant in one classroom but not another, depending on the context. As a result, it is not possible to develop general topic-by-topic or grade levels lists of mathematically significant ideas.

**Pedagogical Opening**

Conscientious teachers continuously seek evidence of their students’ engagement with a wide variety of instructional goals. They take cues from actions big and small, making adjustments and pushing students to elaborate, explain and justify their thinking. Not all student actions, however, are “critical moments” (Walshaw & Anthony, 2008, p. 527) that create “potentially powerful learning opportunities” (Davis, 1997, p. 360). In the interest of
Mathematically Significant Pedagogical Openings To Build On Student Thinking

differentiating student actions that meet this higher threshold, we define *pedagogical openings* as observable student actions that provide an opportunity to work toward an instructional goal.

We build the notion of pedagogical opening from the work of Remillard and Geist (2002), who coined the phrase *openings in the curriculum* in the context of their professional development work with teachers. Remillard and Geist define “openings in the [enacted] curriculum” as instances prompted by learners’ “questions, observations, [and] challenges” (p. 13). Initially they thought of these openings as breaks in facilitators’ plans, but came to view them more as “opportunities for facilitators to foster learning by capitalizing on mathematical or pedagogical issues that arose” (p. 13). For us an opening must be “presented” by students—through observable student actions. To determine whether an opening has been presented one must consider both the *positioning* and the *timing* of an observable student action. These two key criteria are discussed below.

**Positioning.** Building on the notion from the discourse analysis literature in general and the work of Harré (e.g., Davies & Harré, 1990) in particular, we define *positioning* as the way in which observable student actions situate students with respect to an instructional goal. In part, this positioning might be thought of as *how* students are engaged with ideas related to an instructional goal. We argue that students are positioned well with respect to an instructional goal when they engage “deeply” with the content of that goal as opposed to “at a surface level.” Note that such positioning can occur in classrooms for which this kind of engagement is the norm, as well as in classrooms in which a student is struggling to make conceptual sense of an idea that is being presented in a purely procedural manner. In general, evidence of sense-making activity, such as problem solving, reasoning, and making connections, positions students well with respect to related instructional goals and, depending upon the timing, might create openings.
to engage the rest of the class in this same activity. Lack of evidence of these deeper levels of engagement, or evidence of shallow engagement with the content of an instructional goal, does not position students well, as there is little on which to build. The teacher, of course, can actively increase the likelihood of students being engaged in ideas related to an instructional goal by productive teaching moves, such as posing high cognitive demand tasks (Stein et al., 2009).

Positioning, however, is determined by what students do and say, not teachers—consequently, determining positioning requires analyzing observable evidence of student actions.

The constructivist notion of teachers and researchers building “a ‘model’ of the students’ conceptual structures” (von Glasersfeld, 1995, p. 14) helps to illustrate why positioning is a critical criterion of a pedagogical opening. In the context of mathematics classrooms, we speak of teachers building “models of students’ mathematics” (Steffe & Thompson, 2000, p. 268) that can inform instructional decisions. Some student actions provide more evidence of students’ mathematical thinking than others, and certain student actions provide evidence that students are making important connections among mathematical ideas or are bumping into areas where their current mathematical understanding is insufficient. Student actions that provide this last kind of evidence, for example, position students well because they provide evidence that students “see as their own a problem in which their approach is manifestly inadequate” (von Glasersfeld, 1995, p. 15). Instances such as these create opportunities for the teacher to direct classroom student activity toward making sense of the mathematical idea of the instance and thus deepen or expand their current mathematical understandings.

**Timing.** In addition to positioning, identifying a pedagogical opening requires consideration of the timing of observable student actions. In other words, it is not just how students engage with the content of an instructional goal, but also when that engagement takes
place that together determine the presentation of a pedagogical opening. Erickson (2003) referred to teachable moments as “times of opportunity… right times of tactical action” (p. x). He went on to state that “the teacher who knows how to ride the crests of these pedagogical waves with her students is worthy of their trust” (p. x). Anyone who has watched a surfer wait for the perfect wave knows that timing is essential to catching a wave. We contend that for pedagogical openings to occur, the timing must be right to catch a pedagogical wave. And how might someone recognize the “right time”? Whereas good positioning is determined by observable evidence of how a particular student is engaged with the content of an instructional goal, good timing is determined by when this observable evidence takes place with respect to overall instructional goals and the preparation of the class as a whole to engage with the idea being raised. Important moments occur when students are well-positioned at times that are particularly compelling—a reason why such instances are often referred to as teachable moments.

The interplay of positioning and timing. We now use a series of hypothetical examples to illustrate the interplay between positioning and timing in creating pedagogical openings and to demonstrate how one determines whether an instance is a pedagogical opening. Suppose a teacher is presenting a lecture on a particular procedure. Now consider the contrast between students’ actions (in this case, student questions). Student A asks why the procedure works. Student B asks the teacher to repeat their explanation of the last step of the procedure. The contrast we wish to highlight here is that Student A’s action (their “why” question) positions them as trying to make sense of the content at hand—their question provides evidence that they are positioned in a “sense-making” relationship with the mathematics. Student B, on the other hand, may very well have a legitimate reason for asking the teacher to repeat their explanation, but the only evidence their question provides is that they are willing to listen to the teacher’s re-
Mathematically Significant Pedagogical Openings To Build On Student Thinking

explanation. Their action (a “repeat” question) provides no evidence of how they are positioned with respect to the mathematics at hand. Based on these articulated questions, Students A and B position themselves in very different ways with respect to the mathematics. Student A’s question positions them well, whereas Student B’s question does not.

Even though Student A’s question has the potential to present a pedagogical opening, the timing of the question must still be considered. Again, we present two contrasting situations:

1) Student A asks their question in the midst of the teacher’s initial description of the procedure; or
2) Student A asks their question after the teacher has worked through the procedure with several examples. In situation 1, although Student A appears to be situated well with respect the instructional goal, the majority of the class likely is not yet ready to engage with the idea being raised. Because only part of the procedure has been presented, the timing is likely not yet right for discussing why it works. By contrast, in situation 2 the initial instructional goal (describing the procedure) has been completed and the class as a whole has been sufficiently exposed to the procedure that it is reasonable to infer that they are ready to engage in this sense-making activity. Hence the timing is right and Student A’s question in situation 2 presents a pedagogical opening.

Summary. An instance contains a pedagogical opening when it (1) involves an observable student action that positions at least one student well with respect to the content of an instructional goal and (2) the timing of that action is such that the class as a whole can meaningfully engage with that content in ways that support, rather than supplant, overall instructional goals. It is only when both the positioning and timing criteria are met that a pedagogical opening can occur.

The Intersection of Student Thinking, Mathematically Significant, and Pedagogical Opening
Mathematically Significant Pedagogical Openings to build on Student Thinking (MOSTs) occur at the intersection of student thinking, significant mathematics, and pedagogical openings. The relationship among these three characteristics is depicted in Figure 3. In the center region of the diagram, where the three characteristics intersect, observable evidence of student thinking about significant mathematics provides pedagogical openings for working towards the mathematical goals of the classroom, thus creating a MOST. Student actions that frequently fall into this region include those in which students question or comment on a mathematical idea, verbalize their incomplete thoughts as they try to make sense of a mathematical idea, express incorrect mathematical thinking, make an error of substance, or notice a mathematical contradiction. Of course, for these actions to be MOSTs, they must be mathematically significant and provide an opening for working toward a mathematical goal for the class.

Figure 3. Conceptual framework for the relationship among instances that include observable evidence of student thinking, are mathematically significant, and provide pedagogical opportunities.

In identifying MOSTs, we acknowledge that the more complex an understanding one has of students and the ways they interact with content and each other, of mathematics and what it takes for students to learn it, and of goal for students’ mathematical learning and how they might successfully play out in a given day’s lesson, the more likely one is to recognize and make sense
of MOSTs. For example, variations in knowledge and experience make it possible that two observers of the same instance of instruction might have different views about whether it is mathematically significant. Consider the difference between an observer familiar with mathematics education literature on rational numbers, and thus quite knowledgeable about multiplicative reasoning, and one with a limited knowledge of multiplicative reasoning both observing the same lesson. Suppose that, given a problem in which a horse is twice as tall as a dog, students are able to find the height of the horse by multiplying the given height of the dog by 2. Given the height of the horse, these same students divide by $\frac{1}{2}$ to find the dog’s height. The observer who understands the common belief that multiplication makes bigger and division makes smaller would recognize the mathematical significance of this reasoning, while the observer with limited knowledge of multiplicative reasoning might excuse it as a simple error.

Our stance is that although there may be some value-based differences among what various observers consider student thinking, mathematically significant and pedagogical openings, most of the differences, such as in the example above, are instead due to variations in knowledge of mathematics and how children learn it. It is precisely because of this variance that the characteristics of student thinking, mathematically significant and pedagogical opening are useful constructs. Even among observers with differences in knowledge and experience, there is significant overlap in the instances they would identify as MOSTs. In addition, the very act of discussing these differences using a common framework and language provides an opportunity for learning to occur among the observers—both teachers and researchers. The MOST construct provides a lens and a common language for recognizing and agreeing on a critical core of high-leverage mathematical thinking that all teachers can aspire to notice when it occurs in their classrooms.
Putting the Theory into Action

We have found the MOST construct useful in both our research and our teacher education work. When determining whether a MOST has occurred, the focus of our analysis is an “instance”—an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture). In the context of artifact analysis, our unit of analysis is an idea unit (Chi, 1997). Typically an idea unit is one conversational turn or physical expression (such as writing a solution on the board), but it can involve multiple turns. For example, if the expression of an idea were interrupted by another speaker with a comment that merely encouraged the initial speaker (e.g., “yeah,” “okay,” or “um-hum”), the speakers’ initial idea and the continuation of it would be considered a single idea unit.

Regardless of the situation, determining whether an instance qualifies as a MOST involves a systematic analysis of whether the instance embodies the three MOST characteristics. Figure 4 illustrates the analysis sequence we have found to be most effective. Because the construct focuses on productive use of student thinking, we begin the analysis with the foundational characteristic of a MOST—student thinking. Focusing first on this characteristic stems from the perspective that what students say or do during a lesson is critical and should inform the teacher’s actions. We then analyze whether the student thinking is focused on what really matters in mathematics instruction—mathematical ideas. This mathematical analysis of the instance distinguishes our work from more general work on classroom discourse or even “teachable moments” in that we focus on instances that are likely to advance students’ development of mathematical ideas. Collectively, we see these first two steps in the analysis as eliminating certain “noise” from the classroom discourse—those instances that do not involve
both student thinking and significant mathematics. Eliminating these instances allows the teacher or researcher to consider whether an instance creates a pedagogical opening only after it has been determined that the instance might be worth pursuing from the perspective of the disciplinary content. In other words, it does not matter how deeply students are thinking or if the timing of their thinking is right if the thinking is not related to significant mathematics. This sequential analysis of instances keeps the focus on what matters most in a mathematics classroom—working towards goals for students’ mathematical learning.

Figure 4. MOST analysis sequence.

Figure 5 details the flow of analysis through the three characteristics and their associated criteria. The analysis process for determining whether a classroom instance is a MOST begins with questioning whether the instance provides observable evidence of student thinking. If it does not, the analysis ends because the instance cannot be a MOST. If observable evidence of student thinking is present, the mathematics of the instance is then analyzed to determine whether the instance is mathematically significant; that is, whether it satisfies the important mathematics, appropriate mathematics and mathematical goals criteria. As discussed in the

---

1 In using the term “noise” we do not mean to imply that these other instances are either unimportant or detrimental. Many types of interactions are necessary in the creation of quality classroom mathematics discourse. The instances we identify as MOSTs are of particular interest to us because they represent high-leverage opportunities to accomplish the specific goal of building on students’ mathematical ideas.
Mathematically Significant section of the paper, this mathematical analysis takes place linearly; if any mathematics criterion is not met, the analysis ends. The instance is not mathematically significant and therefore not a MOST. If the instance is determined to be mathematically significant, the instance is analyzed in terms of whether the positioning and timing are right to create a pedagogical opening. Again, if either criterion is not met, the analysis ends; if both are met, the instance has met the criteria for all three characteristics and is deemed to be a MOST.

As highlighted above, it is important to note that the mathematical analysis takes place only when observable evidence of student thinking is present, and the pedagogical opening analysis takes place only when there is observable evidence of student thinking about significant mathematics. We have found that taking this flowchart approach to the analysis of an instance brings structure and simplicity to an often chaotic and complex task.
Mathematically Significant Pedagogical Openings To Build On Student Thinking

Figure 5. Analysis process for determining whether a classroom instance is a MOST.
Mathematically Significant Pedagogical Openings To Build On Student Thinking

Analysis of Examples from the Literature

In this section we analyze two teaching episodes from the mathematics education literature to illustrate how the MOST construct can be used as a lens to make more tangible the often abstract but fundamental goal of pursuing students’ mathematical thinking. In doing so, we illustrate how the research team used the analysis approach depicted in the flowchart (Figure 5) to determine whether classroom instances are MOSTs. The first episode (Figure 6) is a 5th grade classroom discussion of fractions led by an expert teacher. The second episode (Figure 7) is a 7th grade classroom discussion about solving equations led by a student teacher. The ST, MS, and PO columns in the figures indicate the three MOST characteristics: student thinking, mathematically significant, and pedagogical opening. A check mark in a column indicates that the characteristic has been met for an instance, while a number in brackets indicates at which criterion of the characteristic the instance failed (see Figure 5 for criteria numbers). Instances with three check marks are those that embody all three characteristics and are MOSTs. (The line numbers and italics in the figures are from the original transcripts.)

**Episode 1: Fractions**

Episode 1 (Figure 6) is taken from Leinhardt and Steele (2005) and comes from a 5th grade class discussion about finding output values for the rule $3x + 1$ given different input values. After the students had demonstrated fluency with large integer values in an input-output table, the teacher, Magdalene Lampert, added $\frac{1}{4}$ as an input and asked for the output. Her initial intent was to focus students on the scope of the rule in preparation for graphing linear functions. The dialogue in Episode 1 diverts from that plan in order to focus on the multiplication and addition of fractions. The transcript begins with a response to the teacher’s request for the output value.
<table>
<thead>
<tr>
<th></th>
<th>Soochow:</th>
<th>T:</th>
<th>MS</th>
<th>PO</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>One and three fourths</td>
<td>How would you explain it please?</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>8</td>
<td>T:</td>
<td>Soochow:</td>
<td>Because one-fourth times three is three-fourths and then you just add 0—add a one.</td>
<td>✓</td>
</tr>
<tr>
<td>9</td>
<td>T:</td>
<td>Okay, so you times by three and then you add one.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>T:</td>
<td>Who can explain why one fourth times three is three fourths? Sun Wu?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Sun Wu:</td>
<td>One fourth, like one fourth of a pie and then somebody brings two more and one times three is three—three pieces of pie that came out of four pieces of pie?</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>12</td>
<td>T:</td>
<td>Okay, are they all the same size? Those three pieces of pie? Lisa?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Lisa:</td>
<td>Yes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>T:</td>
<td>How do you know?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Lisa:</td>
<td>Because if you’re adding one fourth times three you’re going to [—] [—] equal parts</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>T:</td>
<td>Okay, Cause I’m, I’m taking three things that are all the same size. They’re all the size of one fourth, Ali?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Ali:</td>
<td>It could be one fourth [—] could be a whole one.</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>T:</td>
<td>Can you explain what you mean?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Ali:</td>
<td>Can I come to the board?</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>T:</td>
<td>Yes, here take this [chalk] it’s easier to see.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>Ali:</td>
<td>Yes, here take this chalk</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>T:</td>
<td>Um-hum</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>Ali:</td>
<td>And then you could divide it into fourths, four pieces. And then one fourth could be one (points to one segment of circle) and then would be like this one (points to the 1 on the input side of the chart).</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>25</td>
<td>Bridgette:</td>
<td>Me-, he means that if you ha-, if you have one fourth and you make say you color in three of the four pieces [—] equal one whole.</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>26</td>
<td>T:</td>
<td>Is that what you meant?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>Ali:</td>
<td>Yeah.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>T:</td>
<td>Okay, what do you think about that? Ali is saying three times one fourth is one fourth [sic]. Add one fourth and you’d get four so it would be just like here [points to the 4 beside the 1 in the function chart]. But the input number here was one [writes faint 1 in input column beside the 4] and now the input number here is one fourth [points to the ¼ in new chart]. What do you think Sun Wu?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>Sun Wu:</td>
<td>He thinks the um, the one is like one fourth. But it’s really one, another, four.</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>30</td>
<td>T:</td>
<td>What do you think about that Ali? [draws another circle]. How many fourths are there in one whole?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>Ali:</td>
<td>Four fourths [T draws new circle divided into fourths].</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>T:</td>
<td>Four fourths? So if I was going to put a number in here I could put one and a fourth [sic] [writes in column]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>T:</td>
<td>Is there anything I could put in there besides one and a fourth? Elsie?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>Elsie:</td>
<td>Wouldn’t it be one and three fourths?</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>35</td>
<td>T:</td>
<td>Oh, I’m sorry. It should be one and three fourths like that anyway changes chart. Is that what you mean?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>Elsie:</td>
<td>Yeah.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 6. Analyzed transcript excerpt from Leinhardt and Steele (2005, pp. 107-108).
The first instance of student thinking occurs in line 7, when Soochow gives a correct answer. The mathematics of the instance is “when you evaluate 3x+1 at ¼ you get one and three fourths.” This mathematics is closely related to the important mathematical idea of evaluating a linear expression for a fractional input. It is appropriate mathematics because it builds on knowledge available to most 5th grade students, but using fractions in this way is not typically part of their past experience. This idea was in direct service of a goal for the day’s lesson—understanding that a linear equation can be evaluated for non-integer values—so the instance meets all three criteria for being mathematically significant. However, since the correct numerical answer alone does not provide insight into how Soochow arrived at that answer, neither she nor anyone else in the class is positioned well with respect to understanding the idea as a result of the instance. Thus the instance fails to meet the first criterion of a pedagogical opening (as indicated by the [1] in the PO column of Figure 6) and is not a MOST.

In lines 9 and 10 Soochow elaborates on the thinking behind her correct answer. The mathematical idea remains the same, so is mathematically significant for the reasons mentioned above. The difference here, however, is that this response provides evidence that she is well positioned with respect to the goal. In addition, Soochow’s explanation of her thinking is directly related to the problem under discussion, making the timing right to engage all the students in the mathematics of the instance. Thus, the instance meets the criteria for a pedagogical opening, as well as for student thinking and mathematical significance, and is a MOST (as indicated by checkmarks in each of the ST, MS and PO columns). We have here a mathematically significant pedagogical opening to build on student thinking—an opportunity to unpack how students are reasoning about fraction operations in this new context.
In line 12, the discussion shifts to multiplying fractions. Sun Wu’s response in lines 14-16 reflects the thinking that “multiplying one-fourth times three is the same as adding one-fourth three times.” This mathematics is a specific example of the important mathematical idea of operations on rational numbers. Typically, rational number multiplication has been taught by 5th grade, but because students commonly struggle with the idea throughout their school experience (e.g., Moss & Case, 1999; Smith, 1995) the instance meets the appropriate mathematics criterion. Finally, even if the idea were not a goal for the lesson, understanding rational number multiplication is certainly a core mathematical goal for elementary school students, meeting the third and final criterion for being mathematically significant. Su Wu’s explanation provides enough information for us to see that he is positioned well with respect to the goal of understanding rational number multiplication. Given the context of Su Wu’s explanation, the timing is right for engaging all the students in thinking about a critical aspect of rational numbers—the fact that multiplying a fraction by a whole number is similar to multiplying two whole numbers, in that both can be thought of as repeated addition. The instance hence meets the criteria for all three characteristics—student thinking, significant mathematics, and pedagogical opening—and is a MOST.

The student comments by Lisa in line 20 and by Ali in lines 24 and 26 all reflect student thinking and thus meet the first requirement for being a MOST. Lisa’s explanation of “adding one fourth times three” does not provide enough information to be clear what exactly is going on mathematically in her thinking. Because of this lack of clarity, determining whether the mathematics in the instance is related to an important mathematical idea or is merely a misspeaking would require further probing. Ali’s statement in line 24 provides evidence that he is thinking about the size of the pieces, but it is too vague to identify the mathematical idea
behind it without seeking clarification. (Note that although teachers often make assumptions about what students are thinking and fill in gaps in their expressed thoughts, for the purpose of our analysis, we rely on what was audible or visible in determining the mathematics of the instance.) Ali’s request to come to the board in line 26 gives insight into the fact that he is thinking about the need to show what he is thinking, but the instance does not involve a mathematical idea. None of these instances meet the important mathematics criterion for being mathematically significant, rendering them ineligible to be MOSTs.

In line 30 Ali makes a conjecture that the “1” in the numerator of the fraction with the denominator of 4 and the “1” listed as an input value in the table with the corresponding output value of 4 are related to one another. Although the conjecture is faulty, it raises the important mathematical idea of what the values in the numerator and denominator of a fraction represent. This idea and its relationship to the input-output table are accessible to the students, yet something they likely have not yet mastered. It is a precursor to meeting the current classroom goal of making sense of evaluating a linear equation with fraction inputs and is directly connected to the broader goal for these students of having a deep and functional understanding of fractions. Meeting these three criteria of important mathematics, appropriate mathematics, and mathematical goals makes the instance mathematically significant. Ali is clearly engaged in trying to make sense of the situation and is thus well-positioned with respect to significant mathematics. The fact that his comment elaborates on an issue directly related to the conversation at hand makes the timing right as well, so the instance is a MOST.

Bridgette’s restatement of Ali’s idea in lines 36-38 involves the same mathematical idea and thus is mathematically significant for the same reasons. The fact that she builds on Ali’s idea reflects her positioning at the center of the discussion and the timing provides additional “fodder
for [the ongoing] content-related conversation” (Schoenfeld, 2008, p. 57). Meeting the criteria for all three characteristics renders this instance a MOST.

Sun Wu’s comment in lines 47 and 48 reflects his thinking that “Ali’s conjecture about the relationship of the numbers is incorrect.” This thinking is related to the important mathematical practice of “critiqu[ing] the reasoning of others” (NGA-CCSSO, 2010, p. 6). Students at this level are able to engage in this practice, yet are novices, making the mathematics of the instance appropriate for this group of students. This practice is considered an important goal for students at all school levels (e.g., NGA-CCSSO, 2010; NCTM, 2000); therefore, all three criteria are met and the instance is mathematically significant. Sun Wu’s critique positions him in the center of the discussion and the timing provides an opening for Ali to reexamine his thinking and for the whole class to think about the validity of each idea as a means of making sense of the mathematics under discussion. Because it embodies all three characteristics (student thinking, mathematically significant, and pedagogical opening), the instance is a MOST.

In line 51 Ali responded “four fourths” (line 51) to the question of “how many fourths are there in one whole?” (lines 49-50). This statement is similar to Soochow’s response in line 7 in that it reflects student thinking and important mathematics, but it differs on the second mathematically significant criterion of appropriate mathematics. By fifth grade, students typically have mastered the mathematical idea that a whole is made up of the number of parts that it is divided into. As a result, the mathematical idea of the instance is not appropriate mathematics, so the analysis ends and the instance is not a MOST.

Elsie’s correction of the teacher’s error in filling out the input-output chart (line 56) goes back to the important mathematical idea in Soochow’s response in line 7, so is mathematically significant for the same reasons. Elsie is clearly positioned well in that she is following the
Mathematically Significant Pedagogical Openings To Build On Student Thinking

conversation and making sense of the mathematics involved. Although the comment helps to keep the discussion on track, it comes at a time when the idea has already been resolved. Thus the instance fails the timing criterion of pedagogical opening and is not a MOST.

Note that this dialogue also contains several student utterances that fell short of being observable evidence of student thinking (lines 18, 40 and 59). Although these utterances suggest that the students are engaged, they do not provide insight into what they are thinking. They are, however, instances where a teacher might make a move to elicit further information from the students, whose responses, in turn, might generate a MOST. Recall that the teacher cannot generate a MOST—they can only create fertile circumstances for students to do so. It is only when there is observable evidence of student thinking that a MOST can occur.

Episode 1 contained a total of five MOSTs. In each case, the teacher expertly incorporated the MOST into her instruction. Although she responded in multiple ways, in each instance she engaged the class in discussing the important mathematical idea in the instance and used the opportunity to push towards meeting important mathematical goals for the students. We see in her example how MOSTs can be used to support student learning about important mathematics.

**Episode 2: Solving Equations**

The second episode (Figure 7) comes from Blanton, Berenson and Norwood (2001, p. 232). The dialogue occurred during a discussion in a seventh grade general mathematics class. The teacher, a student teacher just beginning her field experience, is leading a discussion about this problem: *Alex had $5 left in his wallet after he spent $12 on snacks and souvenirs at the Jubilee. How much money did he take to the Jubilee?* At the point where the transcript begins, the equation \( m - 12 = 5 \) was on the overhead projector and the teacher had asked the students to
Mathematically Significant Pedagogical Openings To Build On Student Thinking

recall how they isolated the variable in the previous day’s lesson. The responses from the students included “subtract 12 from both sides of the equation,” following a procedure they had used the prior day to solve similar equations that had a plus sign instead of a minus sign.

<table>
<thead>
<tr>
<th>Line</th>
<th>Teacher</th>
<th>Student</th>
<th>MS</th>
<th>PO</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>Teacher: OK, here we already have subtraction (indicating the symbol ‘-’ in ‘m - 12 =5’), so what's the opposite of subtraction?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Students: Addition</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>Teacher: So if I want to make a zero here, what can I do?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Students: Subtract twelve.</td>
<td>✓ ✓ ✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>Teacher: If I subtract twelve, it's going to be a plus minus and a plus minus, so I didn't make a zero. So how am I going to get rid of the subtraction?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>Student: Add</td>
<td>✓ ✓ ✓</td>
<td>[1]</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>Teacher: Addition. So what am I going to add?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>Student: Twelve</td>
<td>✓</td>
<td>[1]</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Teacher: OK, do I add it to one side or 2?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>Student: All sides.</td>
<td>✓ ✓ ✓</td>
<td>[1]</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>Teacher: Both sides. OK, so I have a minus twelve plus twelve, and what is that?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>Students again call out various (inaudible) responses, none of which were zero.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>Teacher: Zero</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>Karl: It's 24. You're supposed to add twelve.</td>
<td>✓ ✓ ✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>Teacher: OK, this is not an addition sign right here (again indicating the symbol ‘-’ in ‘m – 12 =5’). It's a subtraction sign.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>Karl: I know.</td>
<td>✓</td>
<td>[1]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Teacher: If you make it plus the opposite, I have a minus twelve plus a twelve. What is that?</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 7. Analyzed transcript excerpt from Blanton et al. (2001, p. 232).

In their analysis, Blanton et al. (2001) identified the teacher’s “incremental questioning routine” and the clear alignment with the oft-observed Initiate-Respond-Evaluate (Mehan, 1979) discourse pattern. We chose this transcript excerpt to illustrate the analytic value of the MOST construct even when the discourse pattern funnels (Wood, 1998) students toward particular responses. As with the previous example, we illustrate how we use the flowchart (Figure 5) to analyze each instance.

The lesson excerpt begins with students responding “addition” (line 17) to the question, “What’s the opposite of subtraction?” (line 16). This response, although minimal, nevertheless provides observable evidence of student thinking, so meets the criterion for student thinking (as
do all the student comments in Episode 2). The mathematics of the instance is “addition is the opposite of subtraction.” This mathematics is related to the important mathematical idea of recognizing inverse operations. Recognizing these inverse pairs, however, is likely to have been mastered by the students in this 7th grade general mathematics classroom, so the instance fails the second mathematically significant criterion of appropriate mathematics and hence is not a MOST.

The next instance of observable evidence of student thinking occurs in line 19, when, in response to the teacher’s question about making zero in the equation (line 18), students respond with various answers, including “subtract 12.” The mathematics of the instance is “subtracting 12 from the left side of this equation will make a zero.” An important mathematical idea associated with this mathematics is understanding how to make zero pairs in order to isolate variables to solve linear equations. This idea is worth pursuing in general and is accessible to students at this level. Also, given the current discussion and line of reasoning, this mathematical idea is clearly related to mathematical goals for these students (e.g., students will understand that creating zero pairs helps to isolate the variable when solving linear equations). This observable evidence of student thinking is thus mathematically significant. Now, clearly the teacher expected students to respond “add 12” given that all had just been reminded of the fact that addition is the opposite of subtraction. That some students say “subtract 12” in this context provides evidence that they are trying to make sense of the situation (rather than merely parrot back a rote response) and therefore positions students well with respect to the mathematical goal. The timing of this response is also quite compelling—both because of the previously “established” fact that addition is the opposite of subtraction and because “subtract 12” is only one of multiple responses given at this time. This instance thus creates a pedagogical opening
and qualifies as a MOST. We have here a mathematically significant pedagogical opening to build on student thinking—to explore how students are thinking about ‘making zero’ and how that goal might be related to considering inverse operations.

In line 21 students again respond “add,” only this time they respond not to a question of what is the opposite of subtraction, but instead to a question about how they can “get rid of the subtraction” (line 20). The mathematics of this instance is that addition will get rid of subtraction. This mathematics is again related to the mathematical idea that, because addition is the inverse of subtraction, zeroing out the subtraction of a number in an equation requires addition. As previously established, this idea meets all the criteria for being mathematically significant. The observable evidence of student thinking in the instance, however, does not provide evidence of this student making sense of the mathematics at hand. Rather the student seems to have been “coerced” into returning to a statement of the fact that addition is the opposite of subtraction. As such, the student is not positioned well, so the instance fails to be a pedagogical opening and is not a MOST.

In line 23 a student responds “twelve” to the question, “What am I going to add?” (line 22). The mathematics of this instance seems to be, “The number under consideration in the equation is twelve.” Although this mathematical statement is indeed related to the important mathematical idea of using inverse operations to create zero pairs, merely identifying the specific number that is being either added or subtracted is something these students would clearly already be expected to understand. Therefore this mathematics is not appropriate for further consideration, and the instance fails to be mathematically significant and is not a MOST.

The teacher then asks whether 12 should be added “to one side or 2” (line 24) to which a student responds “all sides.” The mathematical idea of this instance is, basically, “What you do
to one side of an equation you need to do to the other side as well.” This situation is quite similar to that of line 21. The mathematical idea of the instance is certainly worth pursuing, accessible to and directly connected to mathematical goals for these students, so the instance is mathematically significant. As in line 21, however, the student seems to have been “set up” for this response—there is no evidence that they are doing anything other than repeating back a conditioned or memorized fact. Because there is no evidence of sense making, the student is not positioned well with respect to the mathematical idea; the instance fails to be a pedagogical opening and therefore is not a MOST.

In line 26 the teacher asks students for the sum “minus twelve plus twelve.” Various incorrect answers are apparently called out, but the transcript provides no clear evidence of student mathematical thinking at this point. The teacher states the correct answer is zero and then in line 28 Karl speaks up: “It’s 24. You’re supposed to add twelve.” The mathematical idea of this instance seems to be that when you add 12 to the 12 in this equation you get 24. Addition of signed numbers is an important mathematical idea, and is accessible but not likely to have been mastered by 7th grade students, so it is also appropriate mathematics. Because this mathematics is clearly related to the mathematical goal previously articulated, the instance is mathematically significant. Karl’s emphatic statement that the answer should be 24 because “you’re supposed to add” is evidence of Karl trying to make sense of this mathematics problem, which positions him well. In addition, Karl is willing to make this claim despite the teacher telling him and the rest of the class that the correct answer should be zero. Karl’s comment is timed just right to create a pedagogical opening to pursue his (and likely others’) conception of this mathematics. This instance is therefore a MOST.
Mathematically Significant Pedagogical Openings To Build On Student Thinking

One final instance of observable evidence of student thinking remains in the transcript when Karl responds with “I know” (line 30) when the teacher reminds him that the sign in front of the 12 in the equation is a subtraction sign, not an addition sign. The mathematics of this instance seems to be exactly what the teacher said, namely, “This symbol [-] is a subtraction sign, not an addition sign.” Although we certainly do want students to know this fact, the mathematical idea of recognizing these symbols is not appropriate, as we would expect these students to have mastered the ideas already. So this instance is not mathematically significant. However, Karl’s statement of “I know” provides even more evidence of his thinking, “that the answer here is 24, regardless of whether addition and subtraction are opposites.” Karl’s willingness to stand by his claim clarifies his positioning as making sense of the mathematics, strengthening the opening his previous comment created. Although not a MOST on its own, we have come to think of instances such as line 30 as a MOST amplification because they provide an opportunity for the teacher to better understand the mathematics of the instance and give the teacher a second chance to capitalize on the MOST that they amplify.

Discussion

We began by describing our conceptualization of the characteristics student thinking, significant mathematics, and pedagogical openings, and the way these characteristics converge to create Mathematically Significant Pedagogical Openings to build on Student Thinking. We then applied the conceptual framework to two examples from the mathematics education literature to help the reader better understand the MOST construct and, in so doing, illustrated how the construct allows us to distinguish high-leverage instances of student thinking from instances that do not embody all three MOST characteristics.
Our analysis of two contrasting episodes demonstrates that MOSTs can occur across grade levels, in different mathematical contexts, in different types of classrooms, and with teachers of varied experiences. In the Fractions episode, where the teacher capitalizes on the MOSTs, the construct highlights the complexity of the teacher's moves and allows us to make some inferences about how she is deciding what to do and when. Here, the MOST construct provides insights into an expert teachers’ skilled practice of building on students’ mathematical thinking. The Solving Equations episode illustrates both that the nature of teachers’ questions influences the likelihood that MOSTs will occur, and that MOSTs can nevertheless occur in any classroom in which students are engaged with mathematical ideas.

Using examples from existing literature also allows us to examine the value added by using the MOST construct as a lens for viewing classroom mathematics discourse. In the remainder of this section we carry out such an examination by contrasting the emphasis of the MOST construct with the emphasis of the theoretical lenses of the original studies from which the episodes were taken. Our intent is not to critique or minimize this related research in any way. Rather, we seek to build on work such as theirs and to explicate just what the MOST construct contributes to the discussion.

Episode 1 was taken from Leinhardt and Steele (2005). Their study analyzed classroom discourse from the perspective of critical features of instructional explanations, using that perspective “to help analyze and systematize the complexity of the classroom discourse” (p. 87). We see their analysis leading to a better understanding of instructional explanations that incorporate student thinking and of the moves that teachers can make to both to create opportunities for MOSTs and to act on them when they occur. Specifically, Leinhardt and Steele described some of the options the teacher had available to her in response to the students’
comments about fractions in Episode 1 and framed her decisions about what comments to pursue in terms of managing dilemmas of teaching (Ball, 1993; Lampert, 2001). The MOST construct further contributes to understanding the dilemmas of teaching by demystifying the process of deciding which student thinking is likely to be the most fruitful to pursue. It does so by providing a framework for identifying pedagogical openings and the student thinking about significant mathematics that leads to them.

To illustrate what the MOST construct adds to the discussion, consider the first few lines of the dialogue in Figure 6. Leinhardt and Steele (2005) described Soochow’s response in line 7 as a correct answer with no apparent need for follow-up—their framework did not provide an explanation for why the teacher chose to pursue it. As described earlier, our analysis points out that, although Soochow’s comment involves significant mathematics, it is not a MOST because it does not position her or the other students well with respect to the mathematics. Because of this lack of positioning, the teacher’s decision to ask Soochow to explain her answer makes perfect sense. Soochow’s comment was not a MOST because it did not include any information about how she was thinking about the mathematics at hand. Soochow’s response in lines 9-10, however, provides that information and thus positions her and the class to engage with the mathematical idea under consideration at that time, making it a MOST. Leinhardt and Steele’s analysis noted that the teacher was able to use the students’ thinking to develop the mathematical ideas; the MOST construct elaborates on how high-leverage student thinking can be identified in ways that provide a roadmap for less expert teachers than Lampert to effectively use student thinking in the development of mathematical ideas, as well as a mechanism for researchers to identify patterns in teachers’ abilities to do the same.
Of note is the way the MOST construct shifts attention from the teacher to the students and provides a concrete way to identify which student thinking is likely to effectively contribute to the instructional explanation. That is not to say that we are not interested in the teacher or that Leinhardt and Steele (2005) were not interested in the students, but rather to highlight the difference in what is in the foreground and background of the two frameworks. They focus on the fact that “particular types of questions and teacher moves prove more fruitful than others in understanding how students are thinking and making sense of the content” (p. 99). In contrast, we focus on the fact that particular instances of student thinking prove more fruitful than others in creating openings to develop important mathematical concepts. One of the reasons that we began the MOST work was because existing research on teachers’ use of student thinking tended to focus on the teachers’ responses to student thinking, rather than on understanding what it was about the thinking that teachers needed to understand to make effective decisions about which thinking was most productive to pursue.

Episode 2 was taken from Blanton et al. (2001). The focus of their study was classroom discourse and the associated emerging practice of a student teacher. Their stated purpose was to look for “linkages between classroom discourse and learning to teach mathematics” (p. 227). In a sense they studied learning to teach through the lens of classroom discourse, looking at episodes over time to try to characterize the evolution of the teacher’s basic discourse routines. We see the MOST construct as a providing a framework to conduct similar research, but related to the specific teaching practice of using students’ mathematical thinking. This construct allows researchers to examine classroom discourse for evidence of both high-leverage student thinking and teachers’ developing abilities to act on it effectively. Rather than analyzing an episode for
the type of discourse that is evident, we examine it to determine to what extent it provides opportunities to enact a particular practice—building on students’ mathematical thinking.

Blanton et al. (2001) found that the early discourse of the novice teacher they studied (represented by the dialogue in Episode 2) was quite “univocal” in nature. Although they do discuss a few of the students’ comments, the main focus of their analysis is on the teacher’s discourse moves. For example, they identified lines such as 16, 18, 20, 22, and 24 as instances of “instructional questions” often associated with univocal discourse (p. 233). Through analyzing the transcript with the MOST construct, we see that students’ answers to these similar questions provide varying opportunities for the teacher to build on student thinking. In fact, the student responses in lines 17, 19, 21, and 23, are each coded differently in our analysis. Lines 23 and 17 fail the MS[1] and MS[2] criteria, respectively, so are not mathematically significant. Line 21 fails the PO[1] criterion, so it is mathematically significant, but fails to create a pedagogical opening. None of these instances would be productive for the teacher to try to build on. Line 19, however, is a MOST, providing an opportunity for the teacher to break out of her univocal discourse pattern and build on students’ mathematical thinking.

Two main points emerge from comparing our analysis with that of Blanton et al. (2001) in this way. First, the MOST construct illuminates the variety of student mathematical thinking that can be elicited despite a univocal discourse pattern. Even though these types of teacher questions are not ideal, they still provide opportunities—although limited—for students to share their mathematical thinking. Second, our framework allows us to identify which instances this teacher could focus on if she decided to try to focus on “using student thinking.” In seeking to change from univocal to dialogic discourse a teacher may overcompensate by attempting to build on every student comment with statements like, “Does everyone agree?” or “Say more.” As a
more effective alternative, the MOST construct provides a means for analyzing which instances have the most potential to be used productively.

Blanton et al. (2001) also identified specific instances in Episode 2 where the teacher did not seem to be focused on students’ mathematics. With respect to line 20, they noted that the teacher was “supplying students with explanations of her own methods” (p. 232), while in line 29, they noted that the teacher “interpreted [Karl’s] utterance (28) as a breakdown in communication” (p. 233) and responded by “explaining her own perspective (29, 31), rather than questioning his utterance” (p. 233). Here the authors identified key instances where the teacher pushed through her own mathematical thinking rather than pursuing students’ thinking. It is possible that the student teacher shared her mathematics with the students because she perceived that something about these instances was mathematically important. Our analysis sheds light on why the instances are important—they were mathematically significant pedagogical openings to build on student thinking. The MOST construct also provides a rationale for using the students’ mathematical thinking in these instances to advance students’ understanding of the mathematics, rather than imposing the teacher’s own mathematics.

Like the focus on discourse in Blanton et al. (2001), the MOST construct is intended to illuminate important aspects of teacher practice. It does so, however, by focusing primarily on students and their mathematics rather than teachers and theirs. This focus allows for detailed analysis of students’ mathematics and whether observable evidence of student thinking provides high-leverage opportunities to build on that thinking. Although we desire the construct to eventually be applied to better understand whether and how teachers have capitalized on MOSTS, part of the power of the construct lies in its focus on the value of the instance separate from a value judgment of the teacher. Focusing on students and their mathematics allows one to
unpack the building blocks of a teacher practice without necessarily critiquing any particular attempt to enact the practice.

Comparing the analyses undertaken by Leinhardt and Steele (2005) and Blanton et al. (2001) with ours illustrates the usefulness of the MOST construct. The analysis process for determining whether a classroom instance is a MOST (Figure 5) zooms in on student thinking, significant mathematics, and pedagogical openings. Thus the MOST construct provides a lens through which both teachers and researchers can better understand instances of student thinking that have the most potential to increase student understanding of important mathematical ideas.

**Conclusion**

Researchers and practitioners in mathematics teacher education advocate the use of student thinking as a means of improving mathematics instruction (e.g., NCTM, 2000). Many teachers we have observed, particularly novices, seem to interpret this call to mean that all student thinking is equally valuable and, consequently, should all be pursued in similar ways. We argue, however, that this interpretation is fundamentally flawed. While teachers certainly need to carefully listen to all student ideas, this listening must be followed by thoughtful consideration of whether a particular idea or comment is worth pursuing in the limited amount of instructional time that is available.

By clearly defining three critical characteristics that distinguish instances that provide high-leverage opportunities to advance students’ mathematical understanding from those that do not, the MOST construct has the potential to become a tool to make sense of classroom interactions. In particular, the construct provides both a means of systematically analyzing instances of classroom discourse and a vocabulary for discussing the mathematical and pedagogical importance of student thinking that arises within such discourse. Thus, the MOST
Mathematically Significant Pedagogical Openings To Build On Student Thinking

Construct can be used to frame (Levin, Hammer, & Coffey, 2009) instances of student thinking in productive ways.

Considering whether an instance embodies the three characteristics of a MOST requires identifying the mathematics in an instance of observable student thinking, as well as the larger mathematical idea to which it is related. Instances that are determined to be mathematical are then framed in terms of both mathematical significance and the pedagogical opening they provide. Engaging in this analysis provides a mechanism for teachers to make informed decisions about what student thinking to pursue, focusing attention on those instances of student thinking that are most likely to advance student learning of mathematics. Framing classroom events in this way has the potential to change the way teachers use student thinking in the classroom. Likewise, the structured analysis process allows teacher educators and researchers to frame teachers’ practice in terms of their use of high-leverage instances of student mathematical thinking. This framing shifts the focus of the work from whether a teacher is using student thinking, to what student thinking a teacher is incorporating into a lesson and why that incorporation is valuable.

Although we acknowledge that mathematics teachers’, teacher educators’ and researchers’ considerations are influenced by a wide range of beliefs about the nature of mathematics and about its teaching and learning, as well as by their own mathematical knowledge (e.g., Ball, Thames, & Phelps, 2008; Maher & Davis, 1990), we present the MOST framework as a mechanism for building mutual recognition and appreciation of high-leverage opportunities to build on students’ mathematical thinking. Engaging in discussions of instances of student thinking using a common language and framework provides an opportunity to advance
understanding of the productive use of student mathematical thinking, and consequently, enhance the teaching and learning of mathematics.

The main purpose of this paper was to articulate the details of the MOST construct. As has been illustrated, the construct provides a lens to examine—and a vocabulary to talk about—instances of student mathematical thinking. This lens also allows researchers to determine whether and to what degree instruction attends to student thinking, significant mathematics, and pedagogical openings. With this lens in place, we believe that subsequent research could be conducted that answers questions such as “How might MOSTs vary across classrooms with diverse populations of students?” and “How can teachers be supported in learning to recognize MOSTs?” Although these questions focus on identifying MOSTs, the intent is that such identification lead to learning to productively use the student mathematical thinking found in MOSTs. Once the MOST construct is used to identify instances of high-leverage student mathematical thinking that occur in classroom dialogue, researchers could study the teacher actions and utterances surrounding these instances. For example, in our analysis of Episode 1, we noticed that there seemed to be patterns in the expert teacher’s responses to instances that were and were not MOSTs. When the instance was a MOST, she typically directed the discussion away from the individual toward the entire class; when an instance involved student thinking, but was not a MOST, she typically asked the student who expressed the thought to explain what they were thinking. Identifying patterns such as these may provide insight into effective teacher moves in response to instances of student thinking. Further research questions include “What teacher moves tend to facilitate or inhibit students’ generation of MOSTs?” and “What are effective ways to capitalize on MOSTs?” This vein of research could contribute to increased
understanding of the nature of instruction that uses student mathematical thinking to support student learning of mathematics.

As we have shown, the MOST construct has potential to contribute to the work of facilitating and researching teachers’ mathematically-productive use of student thinking. Because one of the most important diversities in the classroom is that of student mathematical thinking, a focus on recognizing and using such opportunities for action addresses important classroom equity issues. Teachers’ abilities to provide high quality mathematics education for all students can be enhanced by helping them to identify instances of student thinking about significant mathematics that occur in their classrooms and to recognize when this thinking creates a pedagogical opening to build on that thinking to support student learning. We believe such a lens is significant because MOSTs are high-leverage instances of student thinking that significantly enhance mathematics instruction if incorporated well into classroom lessons. Our conceptualization of MOSTs provides a tool for analyzing practice that can help make more tangible the often abstract but fundamental goal of building on students’ mathematical thinking.

References


Mathematically Significant Pedagogical Openings To Build On Student Thinking


