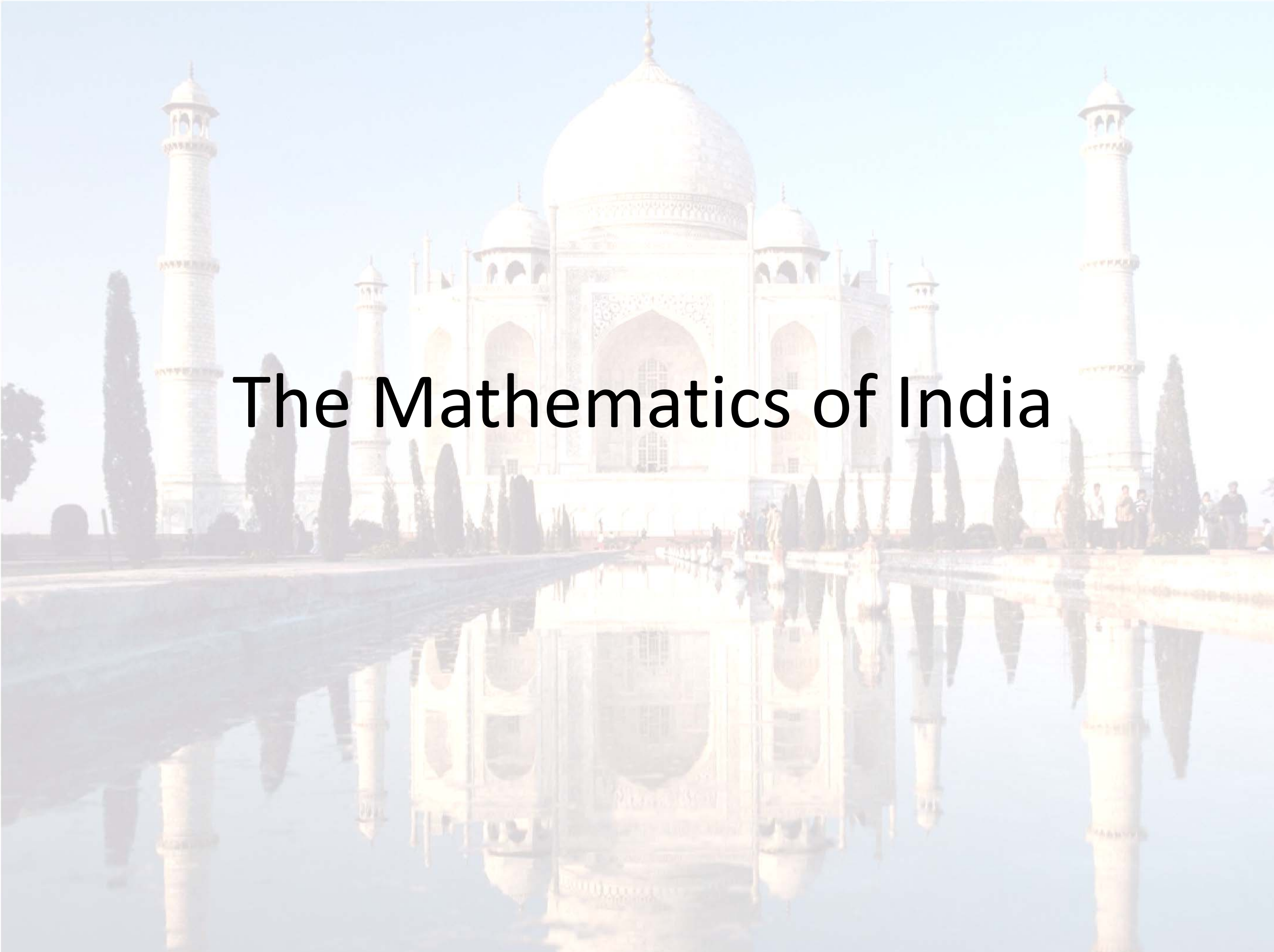
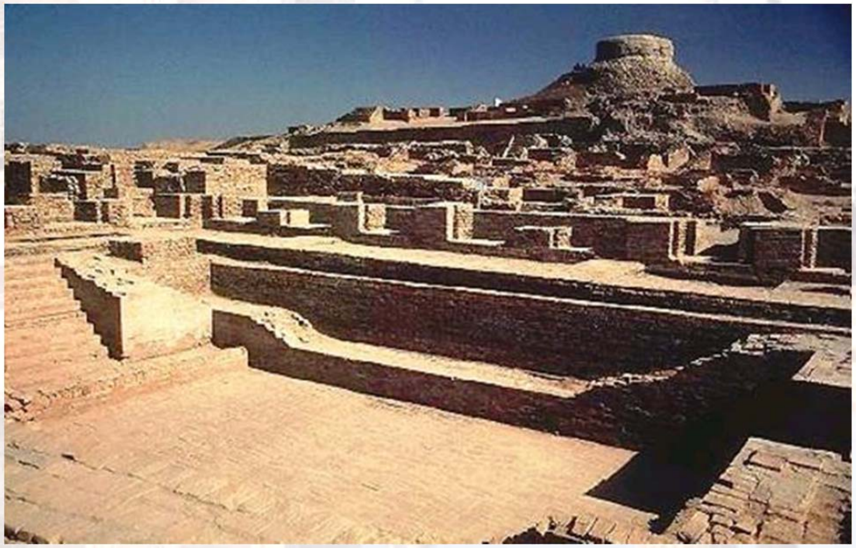


The Mathematics of India

We’ll Discuss 6 Periods

3000 – 1500 BC	Indus Valley Civilization	Weights, measures, brick technology	
1500 – 500 BC	Aryans; Hindu Civilization; Vedas and Upanishads	Vedangas and sulbasutras, Vedic geometry	Baudhayana, Apastamba, Katyayana
500 – 200 BC	Buddhism; Jainism; Mauryan Empire	Vedic and Jaina math; number theory, combinatorics	
200 BC – 400 AD	Triple Division; pervasive influence of Buddhism	Jaina Mathematics: Operations, decimal numeration, algebra, 0.	
400 – 1200 AD	Imperial Guptas	Classical Period of Indian Mathematics	Aryabhata I, Varahamihira, Bhaskara I & II, Brahmagupta, Sridhara, Mahariva
1200 – 1600 AD	Muslim Dynasties; Sikhism; Vijaynagar	Kerala School	

Indus Valley Civilization



Indus Valley Civilization

- Very advanced brick-making technology.
- Over 15 kinds of Harappan bricks have been found, but they all have their dimensions in the ratio of 4:2:1.
- They contain no straw or other binding material, and are still usable today.

Indus Valley Civilization

- Accurate decimal rulers found at Lothal, Harrapa, and Moheno-Daro.
- The Moheno-Daro scale is a fragment of shell 66.2 mm long, with nine carefully sawed, equally spaced parallel lines on average 6.7056 mm apart, and with a mean error of .075 mm.
- Two of the lines are marked with a circular hole and a circular dot. The distance between them is 1.32 inches, or an “Indus inch.”
- A Sumerian *shushi* is exactly half an Indus inch; the Indian *gaz* is 25 Indus inches.

Indus Valley Civilization

- Plumb bobs, or weights, found at Lothal were found to be in decimal relationships.
- If you let a weight of 27.584 grams be “1”, then there were others with values 0.05, 0.1, 0.2, 0.5, 2, 5, 10, 20, 50, 100, 200, and 500.
- In other words, powers of 10, their halves, and their doubles.
- This is pretty much how our money system works (except for the absence of 20-cent pieces and $200 bills).

Indus Valley Civilization

- The language has never been deciphered.
- Much of what we know come from some inscriptions on steatite (soapstone) seals.
- Recent more extensive written records may help.

Indus Valley Civilization

- Summing up: Circumstantial evidence of a wide-spread numerate society.
- *"...In fact the level of mathematical knowledge implied in various geometrical designs, accurate layout of streets and drains and various building constructions etc. was quite high (from a practical point of view).* [R. Gupta]

Aryans and Birth of Hinduism

- The Aryans came from the north and became the dominant people.
- Since the Nazis co-opted the term Aryan, it is common to refer to them as *Indo-Europeans*.
- They spoke what became Sanskrit.
- Tribal, nomadic, war-like.
- Cleared a lot of forests, apparently.

Aryans and the Birth of Hinduism

- They seem to have had a well-developed musical culture, and song and dance dominated their society. Their interest in lyric poetry was unmatched. They loved gambling. They did not, however, have much interest in writing . There are no Aryan writings until the Mauryan period (300 BC).

Aryans and the Birth of Hinduism

- Their religious poetry, the *Vedas*, are the basis for Hinduism.
- Hymns, songs, incantations, etc. to be used by different priests on different occasions.
- Primarily religious. However, "the need to determine the correct times for Vedic ceremonies and the accurate construction of altars led to the development of astronomy and geometry." [R. Gupta]

Mathematics in the Service of Religion

- Various lyric poems from this era, later written down, contain some practical mathematics, now known as *vedic mathematics.*
- Sources include:
  - Vedas
  - Vedangas (Jyotis and Kalpa)
  - Brahmanas (Satapatha)
  - Sulbasutras

Mathematics in the Service of Religion

- Vedangas, or supplements ("limbs") to the Vedas, contained rules for rituals and ceremonies, including the construction of alters. The relevant mathematics included:
  - Use of geometric shapes, including triangles, rectangles, squares, trapezia and circles.
  - Equivalence in terms of numbers and area.
  - Squaring the circle and visa-versa.
  - Early forms of the Pythagorean theorem.
  - Estimations for π.
  - The Rule of Three

Mathematics in the Service of Religion

- The implied estimations for π include:
  - 25/8 (3.125)
  - 900/289 (3.11418685…)
  - 1156/361 (3.202216…)
  - 339/108 (3.1389)

Mathematics in the Service of Religion

- Sulba Sutras: A *Sutra* was a form aiming at brevity. It uses poetic style, avoids verbs, and uses compound nouns. All this renders information easier to memorize in an oral tradition.
- Contained information on the size and shape of altars, both for communal and for private worship.
- Used stakes and marked cords as tools in constructing altars.

Mathematics from the Sulbasutras

- Pythagorean Theorem and Pythagorean triples.
- Finding squares with the same areas as rectangles, the sum or difference of other squares, circles, etc.
- “Circling” a square.
- Doubling or tripling areas of squares, etc.

Mathematics from the Sulbasutras

- Finding irrational square roots:

Given a square of side “1”, “Increase the measure by its third, and this third by its own fourth less the thirty-fourth part of that fourth.”

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}$$

$$= 1.4142156 \ldots$$

Mathematics from the Sulbasutras

- A later commentator, Rama, in the mid-fifteenth century, added two further terms and got seven decimal places of accuracy.

$$-\frac{1}{3 \times 4 \times 34 \times 33} + \frac{1}{3 \times 4 \times 34 \times 34}$$

Mathematics from the Sulbasutras

- A method for calculating square roots by repeated applications of the formula:

$\sqrt{A} = \sqrt{a^2 + r} \approx a + \frac{r}{2a}$ for small *r*.

$$\sqrt{10} = \sqrt{3^2 + 1} \approx 3 + \frac{1}{6} = 3.16666 \ldots \ldots$$

Now $\left(3 + \frac{1}{6}\right)^2 = 10.02777 \ldots$ so let $r = -.02777 \ldots$ and do it again. This yields 3.16228, and $\sqrt{10} = 3.16227766 \ldots$

Jainism

- Jainism prescribes a path of non-violence towards all living beings. Its philosophy and practice emphasize the necessity of self-effort to move the soul towards divine consciousness and liberation. Any soul that has conquered its own inner enemies and achieved the state of supreme being is called *Jina* (Conqueror or Victor).

Jainism

- Every living being has a soul
- Every soul is potentially divine, with innate qualities of infinite knowledge, perception, power and bliss (masked by its karmas).
- Regard every living being as you do yourself, harming no one and being kind to all living beings.
- Every soul is born as a heavenly being, human, sub-human or hellish being according to its own karma.
- Every soul is the architect of its own life, here or hereafter.
- When a soul is freed from karmas, it becomes free and attains divine consciousness, experiencing infinite knowledge, perception, power, and bliss.

Jainism

- Right Faith (right vision), Right Knowledge and Right Conduct (the triple gems of Jainism) provide the way to this realization.
- There is no supreme divine creator, owner, preserver or destroyer. The universe is self-regulated and every soul has the potential to achieve divine consciousness (siddha) through its own efforts.
- Non-violence (to be in soul consciousness rather than body consciousness) is the foundation of right view, the condition of right knowledge and the kernel of right conduct. It leads to a state of being unattached to worldly things and being nonjudgmental and non-violent; this includes compassion and forgiveness in thoughts, words and actions toward all living beings and respecting views of others (non-absolutism).

Jaina Mathematics

- Knowledge of *"Sankhyana"* (i.e., the science of numbers, which included arithmetic and astronomy) was considered to be one of the principal accomplishments of Jain priests.
- Mathematics began to move beyond is uses for practical and religious purposes to an intellectual pursuit for its own sake.

Jaina Mathematics

- Theory of Numbers:
  – Fascination for large $(756 \times 10^{11} \times 8{,}400{,}000^{28})$ and small numbers. (*shirsa)*
  – A *rajju* is the distance traveled by a god in six months if he covers a hundred thousand *yojana* (about a million kilometers) in each blink of his eye.
  – A *palya* is the time it will take to empty a cubic vessel of side one *yojana* filled with the wool of newborn lambs if one strand is removed every century.

Jaina Mathematics

- Infinite sets
  - Enumerable (lowest, intermediate, highest*)
  - Innumerable (nearly, truly, innumerably)
  - Infinite (nearly, truly, infinitely)

*Consider a trough whose diameter is that of the earth….Fill it with white mustard seeds counting one after the other. Similarly fill up with mustard seeds other troughs of the sizes of the various lands and seas. Still the highest enumerable number has not been attained.

Jaina Mathematics

- If and when the highest number N is reached, then you can reach infinity by:
- $N + 1, N = 2, \dots., (N + 1)^2 - 1, (N + 1)^2, \dots$

    and so on.

- This is very much like first infinite ordinal ω, which is followed by the infinite ordinals

$$\omega + 1, \omega + 2 \dots \; \omega + \omega = 2\omega, \dots,$$

$3\omega \dots \omega + \omega = \omega^2, \dots, \omega^{\omega}, \dots$etc.

Jaina Mathematics

- Laws of exponents: $a^{1/2} \times a^{1/4} = \left(a^{1/4}\right)^3$.
- The *ardhacheda* of N is the number of times N can be divided by 2 without remainder. Thus the *ardhacheda* of 32 is 5. Similar terms for 3 and 4. (*trikacheda, caturthacheda*)
- Fractions, simple equations, cubic and quartic equations, sequences and progressions.

Jaina Mathematics

- Combinations and permutations.
  - 63 combinations can be made out of 6 different tastes (bitter, sour, salty, astringent, sweet, hot)
  - Combinations of different meters in poetic compositions, found as coefficients of binomials expansions, leading to
  - Pascal's Chinese-Indian-Muslim-probably-Nephite Triangle! (Although in India, it was used only for these metric poetry studies.)

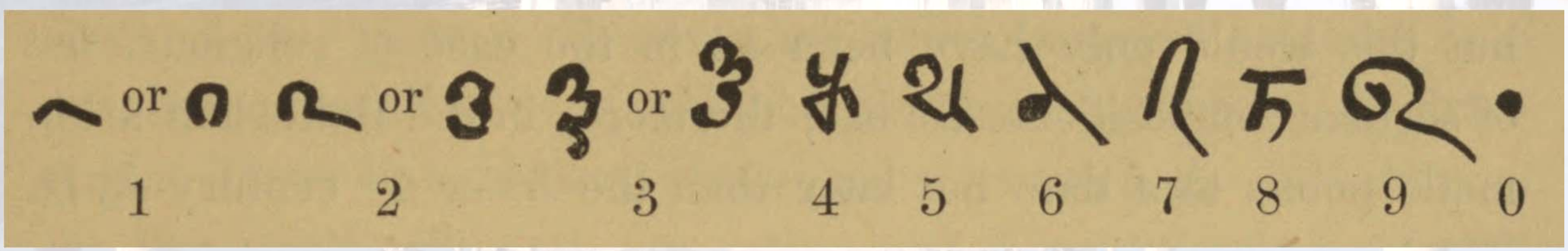
Bakhshali Manuscript

- Unearthed in 1881, written in old Sanskrit, on 70 leaves of birch bark. In pretty bad shape.
- Probably a copy of the original work dating from around the 8th century, and certainly no later than 950 AD. Some prominent historians place the date at pre-450 AD
- *...Capstone of the advance of mathematics from the Vedic age up to that period.*

  [L Gurjar]

Bakhshali Manuscript

- Considerably less concise than other Indian works which were often (if not always) written in a poetic form comprising of short statements of rules, and rarely included examples.



Bakhshali Manuscript

- Examples of the rule of three (and profit and loss and interest).
- Solution of linear equations with as many as five unknowns.
- The solution of the quadratic equation
- Arithmetic (and geometric) progressions.
- Compound Series
- Quadratic indeterminate equations (origin of type $ax/c = y$).
- Simultaneous equations.
- Fractions and other advances in notation including use of zero and negative sign.
- Improved method for calculating square root (and hence approximations for irrational numbers).

Bakhshali Manuscript

- If the equation given is $dn^2 + (2a - d)n - 2s = 0$ Then the solution is found using the equation:

$$n = \frac{-(2a - d) \pm \sqrt{(2a - d)^2 + 8ds}}{2d}$$

(Which is the quadratic equation with $a = d$,

$b = 2a - d$, and $c = 2s$.)

Bakhshali Manuscript

- "Five merchants together buy a jewel. Its price is equal to half the money possessed by the first together with the money possessed by the others, or one-third the money possessed by the second together with the moneys of the others, or one-fourth the money possessed by the third together with the moneys of the others...etc. Find the price of the jewel and the money possessed by each merchant.”

Bakhshali Manuscript

- $x_1/2 + x_2 + x_3 + x_4 + x_5 = p$
  $x_1 + x_2/3 + x_3 + x_4 + x_5 = p$
  $x_1 + x_2 + x_3/4 + x_4 + x_5 = p$
  $x_1 + x_2 + x_3 + x_4/5 + x_5 = p$
  $x_1 + x_2 + x_3 + x_4 + x_5/6 = p$
- If we let $x_1/2 + x_2/3 + x_3/4 + x_4/5 + x_5/6 = q$ the equations become $({}^{377}/_{60})q = p$.
- *Many possible solutions: Indeterminant Analysis.*
- If $q = 60$ then $p = 377$ and $x_1 = 120$, $x_2 = 90$, $x_3 = 80$, $x_4 = 75$ and $x_5 = 72$.

Bakhshali Manuscript

- Two page-boys are attendants of a king. For their services one gets $^{13}/_{6}$ *dinaras* a day and the other $^{3}/_{2}$. The first owes the second 10 *dinaras*. Calculate and tell me when they have equal amounts.
- The solution used the "rule of three:" The lowest common multiple of 2, 6, and 10 is 30. Then using the rule of three, in 30 days the first will have 65, the second 45, and if the debt is paid each have 55.

Bakhshali Manuscript

- The improved method of extracting square roots allowed extremely accurate approximations to be calculated:

$$\sqrt{A} = \sqrt{a^2 + r} \approx a + \frac{r}{2a} - \frac{\left(\frac{r}{2a}\right)^2}{2\left(a + \frac{r}{2a}\right)}$$

- Three examples from the manuscript:

	Bakhshali Method	Modern answer	Correct to:
$\sqrt{487}$	22.068076490965	22.068076490713	9 decimal places
$\sqrt{889}$	29.816105242176	29.8161030317511	5 decimal places*
$\sqrt{339009}$	582.2447938796899	582.2447938796876	11 decimal places

*Using a = 30 instead of a = 29 would get 8 places accuracy.

Road Map:

3000 – 1500 BC	Indus Valley Civilization	Weights, measures, brick technology	
1500 – 500 BC	Aryans; Hindu Civilization; Vedas and Upanishads	Vedangas and sulbasutras, Vedic geometry	Baudhayana, Apastamba, Katyayana
500 – 200 BC	Buddhism; Jainism; Mauryan Empire	Vedic and Jaina math; number theory, combinatorics	
200 BC – 400 AD	Triple Division; pervasive influence of Buddhism	Jaina Mathematics: Operations, decimal numeration, algebra, 0.	Later
400 – 1200 AD	Imperial Guptas	Classical Period of Indian Mathematics	Aryabhata I, Varahamihira, Bhaskara I & II, Brahmagupta, Sridhara, Mahariva
1200 – 1600 AD	Muslim Dynasties; Sikhism; Vijaynagar	Kerala School	

Classical Period - Mathematicians

- Aryabhata I (b. 476)
  - Wrote *Aryabhatiya*, with thirty-three “verses” of mathematics.
    - Arithmetic operations
    - Simple and quadratic equations
    - Indeterminant equations
    - Trigonometry – introduced *sine* and *versine (1-cosine)*.
    - $\pi \approx 3.1416$

Classical Period - Mathematicians

- Brahmagupta (b. 598)
  - Wrote *Brahma Sphuta-siddhanta* (Corrected Siddhanta of Brahma) – an astronomy text, but it included:
    - Mathematical series
    - Some geometry
    - Indeterminant equations
    - Interpolations methods for sine tables
  - Translation of this book made the Islamic, and then the western world, aware of Indian mathematics and astronomy.

Classical Period - Mathematicians

- Sridhara (~ 800)
  - Wrote *Trisatika*, a textbook on arithmetic
    - Elementary operations
    - Extracting square and cube roots
    - Fractions
    - Rules for dealing with 0
    - Summations of arithmetic and geometric series.

Classical Period - Mathematicians

- Mahavira (~ 850)
  - Wrote *Ganita-sara-sangraha*, containing a classification of arithmetic operations and examples.
    - Fractions, including decomposition into unit fractions
    - Operations with 0 and positive and negative quantities
    - Extensions of Jaina work on combinatorics, including general formulas
    - Solutions of various types of quadrics and extensions of work with indeterminant equations
    - Right triangles with rational sides.
  - Recognized his debt to prior mathematicians and their work.

Classical Period - Mathematicians

- Bhaskara II (b. 1114) – "Bhaskara the Teacher"
  - Wrote *Lilavati*, continuing work on arithmetic, combinatorics, operations with 0, etc.
  - Wrote *Bijaganita*, which included:
    - Evaluating surds
    - Simple and quadratic equations
    - Indeterminant equations of higher order
  - Wrote *Siddhanta-siromani*, dealing with trigonometry and containing some proto-infinitesimal calculus ideas.

Classical Period - Mathematics

- Algebra became a distinct subject in its own right.
  - Continued use of symbol for unknown quantity (dot, or letter). Brahmagupta used "*avyakta*" – *invisible or unknown*, shortened to *ya*. But he also called different variables "colors."
  - Use of symbols or abbreviations:
    - 3, 4 *bha* for $3 \div 4$
    - 3, 4 *gu* for 3×4

Classical Period - Mathematics

- Example:

  yava 0 ya 10 *ru* 8

  yava 1 ya 0 *ru* 1

  Here *ya* is the unknown (*yavat tavat*), and *yava* is short for *yavat avad varga*, the square of the unknown; *ru* stands for *rupa*, the constant term.

  So we have $10x + 8 = x^2 + 1$

Classical Period - Mathematics

- Used various versions of the quadratic formula, and a general method not unlike our “completing the square.”
- Used the idea of “algebraic inversion.” This is more like “undoing” things than like “doing the same thing to both sides.”

Classical Period - Mathematics

- O maiden with beaming eyes, tell me, since you understand the method of inversion, what number multiplied by 3, then increased by three-quarters of the product, then divided by 7, then diminished by one-third of the result, then multiplied by itself, then diminished by 52, whose square root is then extracted before 8 is added and then divided by 10, give the final result of 2?

$$[(2)(10) - 8]^2 + 52 = 196;$$

$$\sqrt{196} = 14; \frac{(14)(\frac{3}{2})(7)(\frac{4}{7})}{3} = 28$$

Classical Period - Mathematics

- Indeterminant analysis:
  - Arose in the problems of astronomy, determining orbits of planets.
  - The general problem: Find an integer N which being divided by the given numbers $a_1, a_2, a_3, \ldots, a_m$, leaves remainders $r_1, r_2, r_3, \ldots, r_m$, respectively.
  - Indian mathematicians considered only the case *m*=*2*, while the Chinese worked on larger values of m.

Classical Period - Mathematics

- Brahmagupta and “Pell’s Equation.”
- $ax^2 \pm \mathrm{c} = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	3
1	3

Classical Period - Mathematics

- Brahmagupta and “Pell’s Equation.”
- $ax^2 \pm \mathrm{c} = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	3
1	3
6	

Classical Period - Mathematics

- Brahmagupta and "Pell's Equation."
- $ax^2 \pm \mathrm{c} = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	3
1	3
6	17

Classical Period - Mathematics

- Brahmagupta and "Pell's Equation."
- $ax^2 \pm c = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	**3**
1	3
6	17
35	

Classical Period - Mathematics

- Brahmagupta and “Pell’s Equation.”
- $ax^2 \pm \mathrm{c} = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	3
1	3
6	17
35	99

Classical Period - Mathematics

- Brahmagupta and "Pell's Equation."
- $ax^2 \pm \mathrm{c} = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	3
1	3
6	17
35	99

Classical Period - Mathematics

- Brahmagupta and "Pell's Equation."
- $ax^2 \pm \mathrm{c} = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	**3**
1	3
6	17
35	99
204	

Classical Period - Mathematics

- Brahmagupta and “Pell’s Equation.”
- $ax^2 \pm \mathrm{c} = y^2$
- Example: $8x^2 + 1 = y^2$
  - Start with smallest integral value for x; i.e., 1.
  - Then y = 3
  - Start table:

1	3
1	3
6	17
35	99
204	577

The Trig Functions in India

- In the *Aryabhatiya* of Aryabhata (around 510 AD) the word *ardha-jya* is used for the half-chord.
- Sometimes this is turned around to “chord-half” or *jya-ardha*. Eventually it gets shortened to *jya* or *jiva*.

Sine

- The Hindu word *jya* for the sine was adopted by the Arabs who called the sine *jiba*, a meaningless word with the same sound as *jya*. Now *jiba* became *jaib* in later Arab writings (words in Arabic have mainly consonants, the missing vowels being understood from common usage), and this word does have a meaning, namely a 'fold'.

Sine

- When European authors translated the Arabic mathematical works into Latin they translated *jaib* into the word *sinus* meaning fold in Latin. In particular Fibonacci's use of the term *sinus rectus arcus* soon encouraged the universal use of *sine*.
- The abbreviation *sin*, by the way, was first used by an English minister and professor of Astronomy Edmund Gunter, in the early 1600’s.
- Also *tan*.

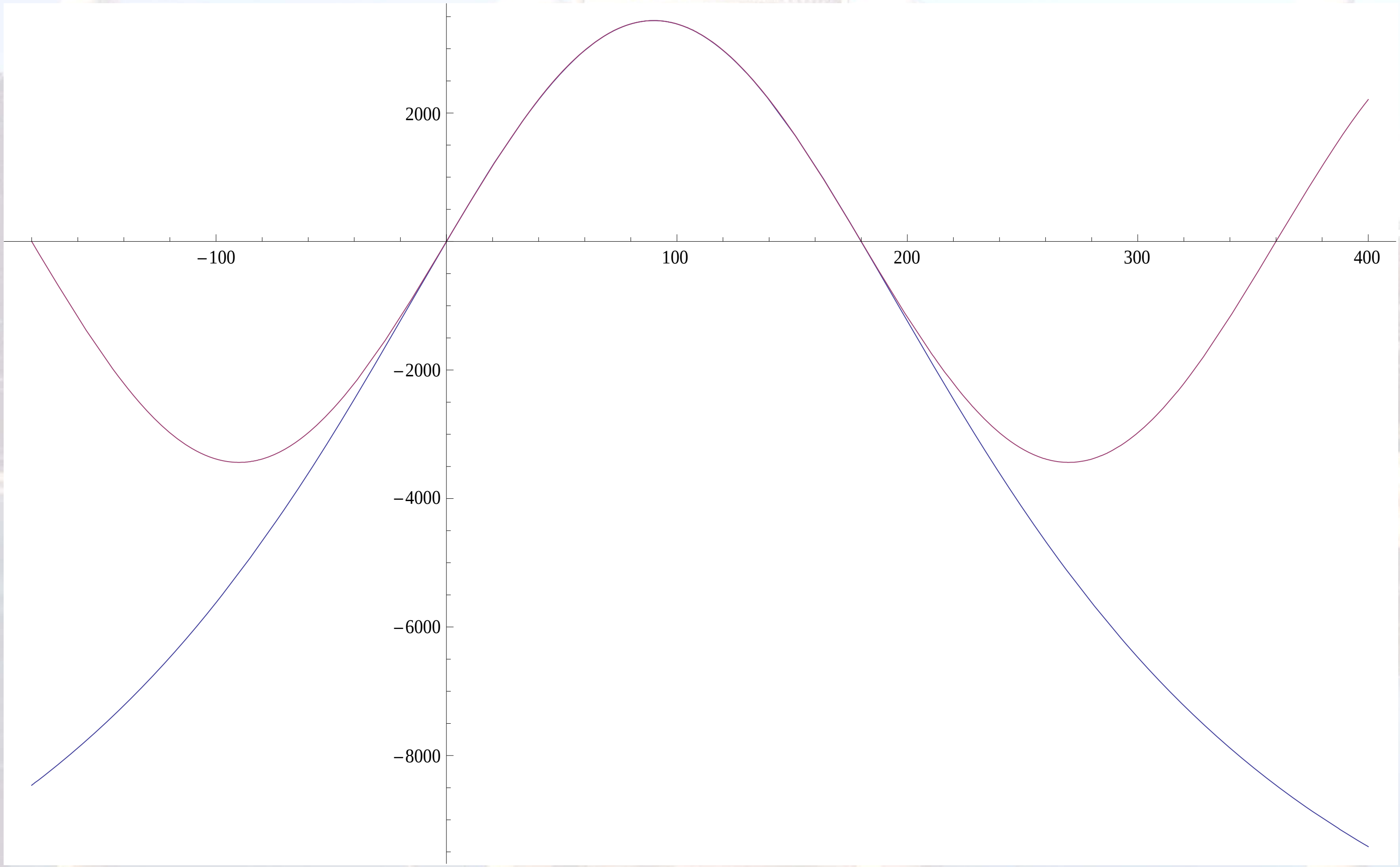
Bhaskara I's Rational Approximation to Sine

- Without any explanation, Bhaskara I provides the following rational function that approximates the sine:

$$\sin x = R \cdot \frac{4x(180 - x)}{40{,}500 - x(180 - x)}$$

where R is the radius of the circle you are using.

A Pretty Good Approximation



The Error: