Circles, Arcs, Inscribed Angles, Power of a Point

Definition: A minor arc is the intersection of a circle with a central angle and its interior. A semicircle is the intersection of a circle with a closed half-plane whose center passes through its center. A major arc is the intersection of a circle with a central angle and its exterior (that is, the complement of a minor arc, plus its endpoints).

Notation: If the endpoints of an arc are A and B, and C is any other point of the arc (which must be used in order to uniquely identify the arc) then we denote such an arc by $\overline{ACB}$.

Measures of Arcs: We define the measure of an arc as follows:

<table>
<thead>
<tr>
<th>Minor Arc $m\overline{ACB} = \mu(\angle AOB)$</th>
<th>Semicircle $m\overline{ACB} = 180$</th>
<th>Major Arc $m\overline{ACB} = 360 - \mu(\angle AOB)$</th>
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Theorem (Additivity of Arc Measure): Suppose arcs $A_1 = \overline{APB}$ and $A_2 = \overline{BQC}$ are any two arcs of a circle with center O having just one point B in common, and such that their union $A_1 \cup A_2 = \overline{ABC}$ is also an arc. Then $m(A_1 \cup A_2) = mA_1 + mA_2$.

The proof is straightforward and tedious, falling into cases depending on whether $\overline{ABC}$ is a minor arc or a semicircle, or a major arc. It is mainly about algebra and offers no new insights. We’ll leave it to others who have boundless energy and time, and who can wring no more entertainment out of “American Idol.”
Lemma: If \( \angle ABC \) is an inscribed of a circle \( O \) and the center of the circle lies on one of its sides, then \( \mu(\angle ABC) = \frac{1}{2} \overarc{AC} \).

\( \square \) WLOG suppose \( O \) lies on side \( BC \), and construct radius \( OA \).

Note that since they are both radii, \( OA = OB \). Thus, triangle \( \triangle AOB \) is isosceles, with \( \mu(\angle ABO) = \mu(\angle BAO) \). By the Exterior Angle Theorem, \( \mu(\angle AOC) = \mu(\angle ABO) + \mu(\angle BAO) = 2 \mu(\angle ABC) \). Thus, \( \mu(\angle ABC) = \frac{1}{2} \mu(\angle AOC) = \frac{1}{2} \overarc{AC} \). \( \square \)

**Theorem (Inscribed Angle Theorem):** The measure of an inscribed angle equals one-half the measure of the arc it intercepts.

\( \square \) Let \( \angle ABC \) be an inscribed angle of a circle \( O \). There are three cases. Case 1: \( O \) lies on \( BC \). This is the case of the above Lemma, which we have already proved.

Case 2: \( O \) is in the interior of \( \angle ABC \). Let \( BD \) be a diameter. Since \( O \) is in the interior, we have angle additivity. Using the Lemma, \( \mu(\angle ABC) = \mu(\angle ABD) + \mu(\angle DBC) \)

\[ = \frac{1}{2} \overarc{AD} + \frac{1}{2} \overarc{DC} = \frac{1}{2} \overarc{AC} \]
Case 3: \( O \) is in the exterior of \( \angle ABC \). Thus either \( BA*BC*BD \) or \( BC*BA*BD \) (the third option, \( BA*BD*BC \), implies \( O \) is interior to \( \angle ABC \)). WLOG, suppose \( BA*BC*BD \). Once again, by angle additivity, the Lemma, and additivity of arcs, we have:

\[
\mu(\angle ABC) = \mu(\angle ABD) - \mu(\angle CBD)
\]

\[
= \frac{1}{2}mA_{ACD} - \frac{1}{2}mA_{CD} = \frac{1}{2}mA_{AC}
\]

**Corollary:** An angle inscribed in a semicircle is a right angle.

**Corollary:** If a quadrilateral \( \square ABCD \) is inscribed in a circle, opposite angles are supplementary.
**Corollary:** An angle whose vertex lies inside a circle and is formed by intersecting chords of the circle (intercepting arcs of measure $x$ and $y$) has measure $\theta = \frac{1}{2} (x + y)$.

\[ \square \text{Note } \mu(\angle ABC) + \mu(\angle BAC) + \mu(\angle ACB) = 180. \text{ Also, } \]
\[ m\widehat{EC} + m\widehat{DB} + x + y = 360. \text{ So, } \]
\[ \frac{1}{2} m\widehat{EC} + \frac{1}{2} m\widehat{DB} + \frac{1}{2} (x + y) = 180. \text{ Now } \]
\[ \mu(\angle ABC) = \frac{1}{2} m\widehat{EC}; \mu(\angle ACB) = \frac{1}{2} m\widehat{DB}, \text{ so } \]
\[ \mu(\angle ABC) + \mu(\angle ACB) + \frac{1}{2} (x + y) = 180 = \mu(\angle ABC) + \mu(\angle ACB) + \mu(\angle BAC) \]
\[ \text{or } \mu(\angle BAC) = \frac{1}{2} (x + y). \square \]
**Corollary:** An angle whose vertex is exterior to a circle and is formed by intersecting secants of the circle (intercepting arcs of measure $x$ and $y$) has measure $\theta = \frac{1}{2}|x + y|$. 

□ First note that

$$m\widehat{DB} + m\widehat{EC} + x + y = 360^\circ; \quad \frac{1}{2}m\widehat{DB} + \frac{1}{2}m\widehat{EC} + \frac{1}{2}x + \frac{1}{2}y = 180^\circ$$

Adding and subtracting a $\frac{1}{2}y$ we get:

$$\frac{1}{2}m\widehat{DB} + \frac{1}{2}m\widehat{EC} + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}y - \frac{1}{2}y = 180^\circ,$$

or

$$\frac{1}{2}m\widehat{DB} + \frac{1}{2}y + \frac{1}{2}m\widehat{EC} + \frac{1}{2}y + \frac{1}{2}x - \frac{1}{2}y = 180^\circ.$$ But,

$$\mu(\angle ABC) = \frac{1}{2}m\widehat{EC} + \frac{1}{2}y,$$ and $$\mu(\angle ACB) = \frac{1}{2}m\widehat{DB} + \frac{1}{2}y.$$ So,

$$\mu(\angle ABC) + \mu(\angle ACB) + \frac{1}{2}(x - y) = 180^\circ,$$ and since

$$\mu(\angle ABC) + \mu(\angle ACB) + \mu(\angle BAC) = 180^\circ,$$ we must have

$$\mu(\angle BAC) = \frac{1}{2}(x - y).$$
Corollary: An angle formed by a chord and tangent of a circle, with its vertex at the point of tangency and intercepting an arc of measure $x$ on that circle, has measure \( \theta = \frac{1}{2} x \).

\[ \text{Diagram of a circle with a chord and tangent.} \]

\[ \text{Diagram showing a diameter AD, and use the fact that } \angle DAB \text{ cuts of the arc } \overarc{DB} \text{ and so has measure } \frac{1}{2} m\overarc{DB}, \text{ and } m\overarc{DB} + x = 180. \text{ The rest follows from the fact that } \mu(\angle BAC) = 90 - \mu(\angle DAB). \]

\[ \text{Diagram showing a diameter AD.} \]
**Theorem (Two Chord Theorem):** When two chords of a circle intersect, the product of the lengths of the segments formed on one chord equals that on the other chord. That is, referring to the figure, \( AP \cdot PB = CP \cdot PD \).

Note that inscribed angles \( \angle ACD \) and \( \angle DBA \) (1 and 2 in the figure) both cut off the arc \( \overline{AD} \) and so are congruent. Also, \( \angle APC \) and \( \angle BPC \) are vertical angles and so are congruent. Thus, by the AA Similarity theorem, \( \triangle ACP \) is similar to \( \triangle DBP \). This immediately gives
\[
\frac{AP}{PD} = \frac{CP}{PB},
\]
and clearing fractions gives the result. \( \blacksquare \)
Theorem (Secant-Tangent Theorem): If a segment $\overline{PA}$ and tangent $\overline{PC}$ meet the circle at the respective points A, B, and C (point of contact) then, referring to the figure, $PC^2 = PA \cdot PB$.

\[ \square \text{Construct segments } \overline{AC} \text{ and } \overline{BC}. \text{ Since } \angle CBA \text{ cuts of arc } \overline{AC} \text{ it has measure } \frac{1}{2} m\overline{AC}. \text{ By Corollary E above, so does } \angle ACP. \text{ Now since } \triangle PBC \text{ and } \triangle PCA \text{ share two congruent angles they are similar under this correspondence. Then } \frac{PC}{PA} = \frac{PB}{PC}, \text{ and clearing fractions gives the result. } \square \]

Corollary (Two-Secant Theorem): If two secants $\overline{PA}$ and $\overline{PC}$ of a circle meet the circle at A, B, C, and D, respectively, then (referring to the figure) $PA \cdot PB = PC \cdot PD$.

\[ \square \text{Draw a tangent from } P \text{ to point } E, \text{ and apply the Secant-Tangent Theorem to both secants:} \]

\[
PA \cdot PB = PE^2 = PC \cdot PD \quad \square
\]
Note that combining the last three results proves that, if P is any point not on a circle, and if P lies on two different secant lines l and m that intersect the circle at points A & B and at points C & D respectively, then \((PA)(PB) = (PC)(PD)\). This number, which is the same for any secant line containing P, is called the power of the point P with respect to the circle. Moreover, if T is a point on the circle and P is external to the circle, \(PT\) is a tangent line, and the \((PT)^2\) is also equal to the power of the point P relative to the circle.