The Saccheri-Legendre Theorem

**Definition:** The *angle sum* for a triangle is the sum of the measures of its three angles. We denote the angle sum of a triangle $\triangle ABC$ as $\sigma(\triangle ABC)$.

Recall a Corollary to the Exterior Angle Inequality that we discussed earlier. We will need this at the very end of the proof:

**Corollary (From earlier notes):** The sum of the measures of any two angles of a triangle is less than 180.

**Saccheri-Legendre Theorem:** The angle sum of a triangle cannot exceed 180.

□ Begin with $\triangle ABC$ and the construction used in the proof of the Exterior Angle Inequality: Find the midpoint $M$ of $BC$ and find $E$ so that $A-M-E$ and $AM = ME$; connect $C$ and $E$ to form $\triangle AEC$. As in the proof of The Exterior Angle Inequality, SAS gives us $\triangle BMA \cong \triangle CME$ and CPCF gives us both

$$\mu(\angle BAE) = \mu(\angle AEC) \text{ and } \mu(\angle BCE) = \mu(\angle ABC). \quad (*)$$

We next prove that this process produces a triangle whose angle sum is the same as the original. By the X Theorem, $B$ is on the $E$ side of $AC$ and the $A$-side of $EC$, so point $B$ is interior to $\angle ACE$. Point $E$ is also interior to $\angle BAC$ (since all of $AE$ is). Thus we have in triangle $\triangle AEC$ that:
\[ \sigma(\triangle AEC) = \mu(\angle EAC) + \mu(\angle AEC) + \mu(\angle ACE) \]
\[ = \mu(\angle EAC) + \mu(\angle AEC) + (\mu(\angle ACB) + \mu(\angle BCE)) \]
\[ = \mu(\angle EAC) + \mu(\angle BAE) + \mu(\angle ACB) + \mu(\angle ABC) \]
\[ \text{(using (*))} \]
\[ = (\mu(\angle EAC) + \mu(\angle BAE)) + \mu(\angle ACB) + \mu(\angle ABC) \]
\[ = \mu(\angle BAC) + \mu(\angle ACB) + \mu(\angle ABC) = \sigma(\triangle ABC) \]

Note that we can repeat the construction on \( \triangle AEC \), finding the midpoint \( N \) of \( EC \) and point \( F \) with \( A-N-F \) and \( AN = FN \), and thus form \( \triangle AFC \). We can continue in this way to form a sequence of triangles \( \triangle ABC, \triangle AEC, \triangle AFC, \triangle AGC, \triangle AHC \), and so forth, that all have equal angle sums. Notice that the construction allows us to use SAS and CPCF to show that the angles at \( E, F, G, H, \) etc are congruent to \( \angle BAM, \angle MAN, \angle NAP \), and so forth. We will call the measures of these angles \( \theta_1, \theta_2, \theta_3, \theta_4, \) etc.

Now the payoff: Assume that the angle sum for \( \triangle ABC \) exceeds 180; that is, suppose \( \mu(\angle BAC) + \mu(\angle ABC) + \mu(\angle ACB) > 180 \). Then
\[ \mu(\angle BAC) + \mu(\angle ABC) + \mu(\angle ACB) = 180 + t \] for some \( t > 0 \). Now, using the construction above, create a succession of triangles \( \triangle AEC, \triangle AFC, \triangle AGC \), etc. all having the same angle sum as \( \triangle ABC \) (namely, \( 180 + t \)). We assert that the measures of angles \( E, F, G, \ldots \) are approaching 0 in the limit.
To prove this, note that \( \theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n = \mu(\angle BAE) + \mu(\angle EAF) + \mu(\angle FAG) + \ldots < \mu(\angle BA = \mu(\angle BAC) \). This is valid since each of E, F, G, H, etc. is interior to \( \angle BAC \) by the X theorem.

Thus the sequence of partial sums \( \sum_{n=1}^{k} \theta_n \) is monotonic increasing and bounded (by \( \mu(\angle BAC) \)). Since this is true, the series \( \sum_{n=1}^{\infty} \theta_n \) converges, and so the sequence \( \{\theta_n\} \) converges to 0.

But this give us what we need: for some \( N \), \( \theta_N < t \). So some triangle, call it \( \triangle ACW \), has angle sum \( 180 + t \) with \( \mu(\angle W) = \theta_N < t \). Then: \( 180 + t = \mu(\angle WAC) + \mu(\angle ACW) + \mu(\angle W) < \mu(\angle WAC) + \mu(\angle ACW) + t \). Cancel \( t \) from both sides to obtain: \( 180 < \mu(\angle WAC) + \mu(\angle ACW) \), a contradiction to the corollary above. ■

**Note:** This is the first of our theorems to really rely heavily on the deeper properties of the real numbers that allow us to find limiting points. We will need these properties again when we prove the continuity of circles, and when we do parallel projection in Euclidean geometry.

**Corollary:** The sum of the measures of two interior angles of a triangle is less than or equal to the measure of their remote exterior angle.

\[
\begin{align*}
\mu(\angle 1) + \mu(\angle 2) + \mu(\angle 3) & \leq 180, \text{ so} \\
\mu(\angle 1) + \mu(\angle 2) + \mu(\angle 3) & \leq \mu(\angle 3) + \mu(\angle 4) \text{ so} \\
\mu(\angle 1) + \mu(\angle 2) & \leq \mu(\angle 4). \quad \blacksquare
\end{align*}
\]

**Note:** The Saccheri-Legendre Theorem and the above corollary are typical of theorems in neutral geometry, in that they give results “close to” Euclidean results but just not quite as “sharp.” In the Euclidean world, the measure of an exterior angle is not only greater than or equal to the sum of the measures of each its remote interiors, its measure is the sum of their measures. We all know that the angle sum of a triangle in Euclidean geometry is not just less than or equal to 180, but in fact equal to 180.
Corollary (Converse to Euclid’s Fifth Postulate): If $l$ and $m$ are two lines cut by a transversal $t$ such that $l$ and $m$ meet on one side of $t$, then the sum of the measures of the two interior angles on that side of $t$ is strictly less than 180.

□ Exercise ■

Because the book will probably refer to this later, we’ll state and prove a helpful little lemma:

Lemma: If $\triangle ABC$ is a triangle and $E$ is a point interior to the segment $BC$, then: $\sigma(\triangle ABE) + \sigma(\triangle ECA) = \sigma(\triangle ABC) + 180$.

□ Note that $E$ is interior to $\angle ABC$, so that we can add angles. Thus $\mu(\angle BAC) = \mu(\angle BAE) + \mu(\angle EAC)$. Moreover, $\angle BEA$ and $\angle CEA$ form a linear pair, so $\mu(\angle BEA) + \mu(\angle CEA) = 180$. Now the sum $\sigma(\triangle ABE) + \sigma(\triangle ECA)$ is the sum of six angles:

$$\mu(\angle ABC) + [\mu(\angle EAB) + \mu(\angle EAC)] + [\mu(\angle BEA) + \mu(\angle AEC)] + \mu(\angle ACB)$$

and using the above we get

$$\sigma(\triangle ABC) + \sigma(\triangle ECA) = \mu(\angle ABE) + [\mu(\angle BAC)] + [180] + \mu(\angle ACB)$$

or in other words, $\sigma(\triangle ABE) + \sigma(\triangle ECA) = \sigma(\triangle ABC) + 180$. ■