Linear Pairs, Vertical Angles, and Supplementary Angles

**Definition:** Two angles $\angle BAD$ and $\angle DAC$ are said to form a *linear pair* if $\overline{AB}$ and $\overline{AC}$ are opposite rays.

**Definition:** Two angles $\angle BAC$ and $\angle EDF$ are said to be *supplementary* or to be *supplements* if their measures add to 180.

Our next theorem relates these two definitions. First we need a lemma.

**Lemma:** If $C*A*B$ and $D$ is in the interior of $\angle BAE$, then $E$ is in the interior of $\angle DAC$. 
**Theorem (The Linear Pair Theorem):** If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then they are supplementary. That is, $\mu(\angle BAD) + \mu(\angle DAC) = 180$. 
As mentioned in the book, and by now to no one’s surprise, this theorem is often taken as an axiom in order to avoid this somewhat messy proof in a high school class. It was taken as an axiom in the book I last used for Math 362, which also avoided the Betweenness Theorem for Rays by making the Protractor Postulate somewhat stronger, as mentioned above.
**Definition:** Two lines $l$ and $m$ are said to be *perpendicular*, written $l \perp m$, if there exists a point $A$ that lies on both $l$ and $m$ and there exists points $B \in l$ and $C \in m$ such that $\angle CAB$ is a right angle.

**Definition:** Let $A$ and $B$ be two distinct points. A *perpendicular bisector* of the segment $\overline{AB}$ is a line $l$ which contains the midpoint of $\overline{AB}$ and such that $l \perp \overline{AB}$.

**Definition:** Angles $\angle BAC$ and $\angle DAE$ form a *vertical pair*, or are *vertical angles*, if rays $\overline{AB}$ and $\overline{AE}$ are opposite, and rays $\overline{AC}$ and $\overline{AD}$ are opposite, or rays $\overline{AB}$ and $\overline{AD}$ are opposite and $\overline{AC}$ and $\overline{AE}$ are opposite.

There are three easy theorems to prove, all of which make good exercises.

**Theorem:** If line $l$ and $m$ are perpendicular, then they contain rays that make four different right angles.

**Theorem:** If $A$ and $B$ are two distinct points, then there exists a perpendicular bisector for $\overline{AB}$.

**Theorem:** Vertical angles are congruent.

The book has a final theorem, the **Continuity Axiom** (Guess why it’s called an axiom? Right! Someone else used it that way!), which we won’t need until we prove some things about circles later. If we need it then (we may or may not use the book’s proof), we’ll deal with it then.
Independence Issues for our First Five Postulates

Our first two axioms were the Existence and Incidence Postulates. It is not hard to see that the Incidence Postulate is independent of Existence. A model with two points and no lines satisfies Existence but not Incidence; the Three-Point Geometry satisfies both.

It is also fairly easy to see that the Ruler Postulate is independent of the first two: whereas the Cartesian plane with either the Euclidean or Taxicab Metric satisfies all three, the Three-Point Geometry satisfies only the first two.

That Plane Separation is independent of our first three postulates can be demonstrated with the Missing-Strip Incidence Plane. This is the Cartesian Plane with a strip removed:

Points: \{(x,y) | x, y \in \mathbb{R}, and x \leq 1 or x > 2\}

Lines: \{(x,y) | x, y \in \mathbb{R}, and x \leq 1 or x > 2 and ax + by +c =0 for some a, b, c \in \mathbb{R}, a, b not both 0\}

Which brings us to our fifth postulate, the Protractor Postulate. In fact, it is not independent of the other postulates. That is, we could in fact create protractor coordinates using only results proved from our first four postulates. We prove this by showing that, given any model of the first four axioms, we can build into that model a system of protractor coordinates that satisfies the Protractor Postulate. Thus there are no models of the first four axioms that are not also models of the fifth.