Inequalities for Triangles and Pointwise Characterizations

**Theorem (The Scalene Inequality):** If one side of a triangle has greater length than another side, then the angle opposite the longer side has the greatest measure, and conversely.

\[\square\text{Referring to the diagram, let } AB > AC\text{ and find } D\text{ such that } A*D*B\text{ and } AD = AC.\text{ Since } D\text{ is interior to } \angle ACB, \text{ we have } \mu(\angle ACB) > \mu(\angle 1) = \mu(\angle 2). \text{ Since } \mu(\angle 2) > \mu(\angle B) \text{ by the exterior angle inequality, we have } \mu(\angle ACB) > \mu(\angle B).\]

For the converse, suppose that \(m\angle C > m\angle B\). There are three possibilities for the relationship between \(AB\) and \(AC\): Either \(AB < AC\), \(AB = AC\), or \(AB > AC\). By what we just proved, we cannot have \(AB < AC\) or else \(m\angle C < m\angle B\), a contradiction. Moreover, if \(AB = AC\), the triangle is isosceles and \(m\angle C = m\angle B\), a contradiction. So, \(AB > AC\). \(\blacksquare\)

**Corollary 1:** If a triangle has an obtuse or right angle, then the side opposite that angle has the greatest length.

**Definition:** A triangle is a *right* triangle if it has a right angle. The side opposite the right angle is called the *hypotenuse* and the other two sides are called *legs*.

**Corollary 2:** In a right triangle, the hypotenuse has length greater than that of either leg.
Theorem (The Triangle Inequality): In any triangle, the sum of the measures of two sides is greater than that of the third side. More generally: For any three distinct points $A$, $B$, and $C$, $AB + BC \geq AC$, with equality if and only if $A*B*C$.

Case 1: $A$, $B$, and $C$ are not collinear: Given $\triangle ABC$, extend $BC$ to point $D$ so that $C*B*D$ and $BD = AB$. Then $DC = DB + BC = AB + BC$ since $C*B*D$.

Since $\triangle ADB$ is isosceles, $\mu(\angle 1) = \mu(\angle 2)$, and point $B$ is interior to $\angle DAC$. So $\mu(\angle DAC) = \mu(\angle 2) + \mu(\angle 3) > \mu(\angle 2) = \mu(\angle 1) = \mu(\angle ADC)$. By the Scalene Inequality, $DC > AC$, so $AB + BC > AC$.


Note: We have shown that if $A*B*C$, $AB + BC = AC$, and if not $A*B*C$, then $AB + BC > AC$. This establishes the “if and only if” statement. ■
Corollary (Median Inequality – Not in our Text): Suppose that $AM$ is the median to side $BC$ of $\triangle ABC$ (i.e., $M$ is the midpoint of $BC$). Then $AM < \frac{1}{2}(AB + AC)$.

Find point $D$ such that $M$ is the midpoint of $AM$. Then by SAS and CPCF, $AB = CD$. Consider $\triangle ACD$. By the triangle inequality, $AD < AC + CD = AC + AB$. But $AD = 2AM$, so we have $2AM < AB + AC$, or $AM < \frac{1}{2}(AB + AC)$.  ■
Theorem (SAS Inequality, Alligator Theorem or Hinge Theorem): If in \( \triangle ABC \) and \( \triangle XYZ \) we have \( AB = XY \), \( AC = XZ \), but \( \mu(\angle A) > \mu(\angle X) \), then \( BC > YZ \), and conversely, if \( BC > YZ \), then \( \mu(\angle A) > \mu(\angle X) \).

\[ \square \] Construct ray \( \overrightarrow{AD} \) between \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) with \( \mu(\angle BAD) = \mu(\angle X) \), and with \( AD = XZ = AC \). Then \( \triangle ABD = \triangle XYZ \) by SAS, and \( BD = YZ \) by CPCF.

As in the figure below, construct the bisector of \( \angle DAC \). This cuts segment \( BC \) at an interior point \( E \). (Why?) Then \( \angle DAE = \angle EAC \), \( AE = AE \), and \( AD = AC \), so \( \triangle DAE = \triangle CAE \) by SAS. Then \( DE = EC \), and employing the triangle inequality on \( \triangle BED \), and because \( B*E*C \), \( BC = BE + EC = BE + DE > BD = YZ \), so \( BC > YZ \).

For the converse, use the same trick as in the Scalene Inequality: Suppose \( BC > YZ \) but \( \mu(\angle A) \leq \mu(\angle X) \). If \( \mu(\angle A) = \mu(\angle X) \), then \( BC = YZ \) by SAS and CPCF. If \( \mu(\angle A) < \mu(\angle X) \), then the proof we just gave would establish \( BC < YZ \), a contradiction. So \( \mu(\angle A) > \mu(\angle X) \). \( \blacksquare \)
**Theorem:** If $l$ is a line and $P$ is a point not on $l$, and let $F$ be the foot of the perpendicular from $P$ to $l$ (i.e., the point where the perpendicular to $l$ that contains $P$ intersects $l$). If $R$ is any point of $l$, then $PR > PF$.

□ Immediate from the above corollary that the hypotenuse of a right triangle is longer than either leg. ■

**Definition:** If $l$ is a line and $P$ is a point not on $l$, the *distance from $P$ to $l$* is the distance from $P$ to the foot $F$ of the perpendicular from $P$ to $l$.

**Theorem (Pointwise Characterization of the Angle Bisector):** Let $A$, $B$, and $C$ be three noncollinear points and let $P$ be a point in the interior of $\angle BAC$. Then $P$ lies on the angle bisector of $\angle BAC$ if and only if $P$ is equidistant from the sides of the angle, i.e., the lines $\overline{AB}$ and $\overline{AC}$.

**Theorem (Pointwise Characterization of the Perpendicular Bisector):** The set of all points equidistant from each of two points $A$ and $B$ is the perpendicular bisector of $\overline{AB}$.

The proofs of these two theorems are straightforward applications of isosceles triangles and congruence theorems, and make good exercises.
Theorem (The Continuity of Distance): Given ray \( \overline{AB} \) and any point \( O \) not on \( \overline{AB} \), define a function \( d(x) \) for any real \( x \geq 0 \) as the distance from \( O \) to \( P \) on \( \overline{AB} \), where \( x \) is the distance from \( A \) to \( P \). That is, \( d(x) = OP \iff x = AP \). Then \( d(x) \) is continuous.

\[ \square \] Pick \( \varepsilon > 0 \). Let \( R \) and \( S \) be points of \( \overline{AB} \) such that \( AR = x \) and \( AS = y \). Let \( |y - x| = |AS - AR| = RS < \varepsilon \). By the Triangle Inequality, \( OR \leq OS + RS \) and \( OS \leq OR + RS \).

Either way, \( |OS - OR| \leq RS \). Then we have \( |d(y) - d(x)| = |OS - OR| \leq RS < \varepsilon \). Thus \( d(x) \) is continuous. \( \blacksquare \)

Note: We will use this theorem later when we prove the Elementary Continuity of Circles in Chapter 10.