An **axiom system** includes:

- undefined terms, and
- axioms, or statements about those terms, taken to be true without proof.

A **model** for an axiom system is a mathematical system in which:

- every undefined term has a specific meaning in that system, and
- all the axioms are true.

Example of an axiom system and a model:

Undefined terms: member, committee, on.

### Axiom 1:
Every committee has exactly two members on it.

### Axiom 2:
Every member is on at least two committees.

**Model 1:**

**Members:** Joan, Anne, Blair  
**Committees:**  
{Joan, Anne}  
{Joan, Blair}  
{Anne, Blair}  
**On:** Belonging to the set.

**Model 2:**

**Members:** Joan, Anne, Blair, Lacey  
**Committees:**  
{Joan, Anne}  
{Joan, Blair}  
{Joan, Lacey}  
{Anne, Blair}  
{Anne, Lacey}  
{Blair, Lacey}  
**On:** Belonging to the set.

Some terminology about statements and axioms:

A statement $P$ is **independent** of axioms $A_1, A_2, \ldots, A_n$ provided $P$ cannot be proved from $A_1, A_2, \ldots, A_n$. We can show that a statement $P$ is independent of a set of axioms by showing that there is a model of the axioms in which $P$ is false.

A statement $P$ is **undecidable** in a system of axioms $A_1, A_2, \ldots, A_n$ provided neither $P$ nor $\neg P$ can be proved from $A_1, A_2, \ldots, A_n$; in other words, both $P$ and $\neg P$ are independent of the axioms. We can show a statement $P$ is undecidable in an axiom system by showing there is a model of the axioms in which $P$ is false, and also a model of the axioms in which $P$ is true.
Axiom systems ought to be:

- **Consistent**, that is, free from contradictions. This is true provided there is a model for the system. If so, we know we cannot prove a contradiction through logical reasoning from the axioms.

- **Independent**, so that every axiom is independent of the others. Thus, each axiom is essential and cannot be proved from the others. This can be demonstrated using a series of models.

In addition, axiom systems can be:

- **Complete**, so that any additional statement appended as an axiom to the system is either redundant (already provable from the axioms) or inconsistent (so its negation is provable).

- **Categorical**, so that all models for the system are isomorphic, i.e., exactly the same except for renaming.

An example of a categorical axiom system:

Undefined terms: point, line, on.

Axiom 1: There exist exactly three points.

Axiom 2: Each two points are on exactly one line.

Axiom 3: Each two distinct points have exactly one point on both of them (i.e., common to both)

Axiom 4: Not all points are on the same line.

This is also an example of a complete axiom system, because

**Theorem**: Any categorical axiom system is complete.

Outline of proof: We proceed by contradiction. Let $A = \{A_1, A_2, \ldots, A_n\}$ be an axiom system. If it is not complete, there is a statement $P$ that is neither inconsistent nor redundant. So, $P$ is consistent with $A$, and $P$ is not provable from $A$. So, there is a model of $A$ in which $P$ is true, and a model of $A$ in which $\neg P$ is true. These two models must be different.
By the way, one of the great results of modern mathematical logic is that any consistent mathematical system rich enough to develop regular old arithmetic will have undecidable statements. Thus, no such axiom system can be complete. This was proved by Kurt Gödel in 1931.

This demonstrates an inherent limitation of axiom systems; there is no set of axioms from which everything can be deduced. There are a number of philosophical debates surrounding this issue.

A geometric example of undecidable statements:

Axioms:
1. There exist exactly six points.
2. Each line is a set of exactly two points.
3. Each point lies on at least three lines.

“Theorem:” Each point lies on exactly three lines.

“Theorem:” There is a point which lies on more than three lines.

3’. Each point lies on exactly three lines.

3*. Five of the six points lie on exactly three lines, and the sixth lies on more than three lines.

3#. Each point lies on exactly four lines.

In our work in geometry, we will establish axiom an axiom system a little at a time. Occasionally, we will stop to consider whether the axiom we are about to add is in fact independent of the axioms we have established so far. Thus, we will have to have some facility in creating models.

Some points to remember:

♦ Good old naive set theory underlies everything we do in this class--that is, we understand sets, elements, belonging to sets, etc. Lines and planes are sets of points. So saying a point “lies on” a line means that the point belongs to the line as a set of points.

♦ We have a model of Euclidean geometry already: the set of all ordered triples of real numbers, with the usual definition of lines, planes, distance, etc. as built in analytic geometry. It will always be a model of our axiom system, but not the only model.